EXOTIC HOLONOMY ON MODULI SPACES OF RATIONAL CURVES

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ABSTRACT.

Bryant [Br] proved the existence of torsion free connections with exotic holonomy, i.e. with holonomy that does not occur on the classical list of Berger [Ber]. These connections occur on moduli spaces $\mathcal{Y}$ of rational contact curves in a contact threefold $\mathcal{W}$. Therefore, they are naturally contained in the moduli space $\mathcal{Z}$ of all rational curves in $\mathcal{W}$.

We construct a connection on $\mathcal{Z}$ whose restriction to $\mathcal{Y}$ is torsion free. However, the connection on $\mathcal{Z}$ has torsion unless both $\mathcal{Y}$ and $\mathcal{Z}$ are flat.

We also show the existence of a new exotic holonomy which is a certain sixdimensional representation of $Sl(2,\mathbb{C}) \times Sl(2,\mathbb{C})$. We show that every regular H_3-connection (cf. [Br]) is the restriction of a unique connection with this holonomy.

§0 Introduction.

Since its introduction by Èlie Cartan, the *holonomy* of a connection has played an important role in differential geometry. Most of the classical results are concerned with the holonomy of Levi Civita connections of Riemannian metrics. In 1955, Berger [Ber] classified the possible irreducible Riemannian holonomies and much work has been done since to study these holonomies and their applications. See [Bes] and [Sa] for a historical survey and also [J] for more recent results.

At the same time, Berger also partially classified the possible non-Riemannian holonomies of torsion free connections. However, his classification omits a finite number of possibilities, which are referred to as *exotic holonomies*. As of yet, the complete list of exotic holonomies is still not known.

The incompleteness of Berger's list and therefore the existence of exotic holonomies was shown by Bryant [Br]. He investigated the irreducible representations of $Sl(2,\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For each $d \geq 1$, we can regard $Sl(2,\mathbb{F})$ as a subgroup $H_d \subseteq Gl(d+1,\mathbb{F})$ via the (unique) $(d+1)$-dimensional irreducible representation of $Sl(2,\mathbb{F})$ which will be described below. Moreover, if we let $G_d \subseteq Gl(d+1,\mathbb{F})$ be the centralizer of H_d, then G_d may be regarded as a representation of $Gl(2,\mathbb{F})$. For $d \geq 3$, these representations do not occur on Berger's list of possible holonomies and are therefore candidates for exotic holonomies.

In his paper, Bryant showed that in the case $d = 3$ torsion free connections with holonomies H_3 and G_3 do exist both if $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$. We shall refer to them

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as H_3-connections (G_3-connections respectively). From now on, we shall assume that $\mathbb{F} = \mathbb{C}$ unless stated otherwise.

G_3-structures occur naturally in the following way: let $\mathcal{W}$ be a complex contact threefold and suppose there is a rational contact curve C in $\mathcal{W}$ such that the restriction of the contact line bundle $L|_C$ has degree -3. Then the moduli space $\mathcal{Y}$ of all close-by contact curves carries a torsion free G_3-connection.

Conversely, every holomorphic torsion free G_3-connection is locally equivalent to a connection on such a moduli space $\mathcal{Y}$ [Br].

Before we proceed, let us briefly describe the irreducible representations of $Sl(2,\mathbb{C})$, $Gl(2,\mathbb{C})$ and $Gl(2,\mathbb{C}) \times Sl(2,\mathbb{C})$.

For $n \in \mathbb{N}$, let $\mathcal{V}_n \subseteq \mathbb{C}[x,y]$ be the $(n+1)$-dimensional subspace of homogeneous polynomials of degree n. There is an $Sl(2,\mathbb{C})$-action on $\mathcal{V}_n$ induced by the transposed action of $Sl(2,\mathbb{C})$ on $\mathbb{C}^2$, i.e. if $p \in \mathcal{V}_n$ and $A \in Sl(2,\mathbb{C})$ then

$$(A \cdot p)(x,y) := p(u,v) \quad \text{with} \quad (u,v) = (x,y)A.$$

Of course, this formula also describes an action of $Gl(2,\mathbb{C})$ on $\mathcal{V}_n$.

The irreducible representations of $Gl(2,\mathbb{C}) \times Sl(2,\mathbb{C})$ can be described as follows: for $n, m \in \mathbb{N}$, we let $\mathcal{V}_{n,m} := \mathcal{V}_n \otimes \mathcal{V}_m$. Then the action of $Gl(2,\mathbb{C}) \times Sl(2,\mathbb{C})$ on $\mathcal{V}_{n,m}$ is defined by

$$(A,B) \cdot (p \otimes q) := (A \cdot p) \otimes (B \cdot q)$$

with the actions of $Gl(2,\mathbb{C})$ and $Sl(2,\mathbb{C})$ on $\mathcal{V}_n$ and $\mathcal{V}_m$ from above. We define the subgroup $G_{n,m} \subseteq Gl(\mathcal{V}_{n,m})$ to be the image of this representation. Also, we let $H_{n,m} \subseteq G_{n,m}$ be the image of $Sl(2,\mathbb{C}) \times Sl(2,\mathbb{C}) \subseteq Gl(2,\mathbb{C}) \times Sl(2,\mathbb{C})$. In other words, $H_{n,m} = G_{n,m} \cap Sl(\mathcal{V}_{n,m})$.

It is well known [H] that these are complete lists of the irreducible representations of $Sl(2,\mathbb{C})$, $Gl(2,\mathbb{C})$ and $Gl(2,\mathbb{C}) \times Sl(2,\mathbb{C})$ respectively.

Given a rational contact curve C in $\mathcal{W}$ as above, it turns out that its normal bundle $N_C \to C$ is equivalent to $\mathcal{O}(2) \oplus \mathcal{O}(2)$. By Kodaira's Deformation Theorem [K], the moduli space $\mathcal{Z}$ of *all* curves near C is a smooth analytic manifold. Obviously, $\mathcal{Y} \subseteq \mathcal{Z}$.

The tangent space $T_C\mathcal{Z}$ can be identified with $H^0(\mathcal{O}(2) \oplus \mathcal{O}(2)) \cong \mathcal{V}_{1,2}$ in a natural way. Therefore, $\mathcal{Z}$ carries a canonical $G_{1,2}$-structure.

The main objective of this paper is to investigate the correlation between the G_3-structure on $\mathcal{Y}$ and the $G_{1,2}$-structure on $\mathcal{Z}$. It had been conjectured in [Br] that the latter structure is torsion free. However, we show that almost the exact opposite is true. Namely, we shall prove

Theorem 0.1. *Let $\mathcal{W}$ be a complex contact threefold, let C be a rational contact curve in $\mathcal{W}$ such that the restriction of the contact line bundle $L|_C$ has degree -3, and let $\mathcal{Z}$ ($\mathcal{Y}$ respectively) be the moduli space of rational curves (rational contact curves respectively) in $\mathcal{W}$ close to C. Then the canonical $G_{1,2}$-structure on $\mathcal{Z}$ is torsion free if and only if the G_3-connection on $\mathcal{Y}$ is flat.*

This means that we cannot in general expect the $G_{1,2}$-structure on $\mathcal{Z}$ to be torsion free. However, we can make some statement about its torsion.

Theorem 0.2. *Let $\mathcal{W}$ be a complex threefold, let C be a rational curve in $\mathcal{W}$ whose normal bundle is equivalent to $\mathcal{O}(2) \oplus \mathcal{O}(2)$ and let $\mathcal{Z}$ be the moduli space of curves in $\mathcal{W}$ close to C, equipped with the canonical $G_{1,2}$-structure. Then there is a subbundle $T \subseteq \Lambda^2 T^*\mathcal{Z} \otimes T\mathcal{Z}$ of rank four and a unique $G_{1,2}$-connection on $\mathcal{Z}$ whose torsion is a section of T.*

The point of Theorem 0.2. is that the rank of $\Lambda^2 T^*\mathcal{Z} \otimes T\mathcal{Z}$ equals 90, so T has a large codimension. In other words, Theorem 0.2. states that the $G_{1,2}$-structure on $\mathcal{Z}$ has "very little torsion".

Theorem 0.1. raises the questions if there are any non-flat torsion free $G_{1,2}$-structures at all.

Theorem 0.3.

(1) *The holonomy of a torsion free $G_{1,2}$-connection is contained in $H_{1,2}$. Thus, every torsion free $G_{1,2}$-structure admits a (one-parameter family of) $H_{1,2}$-reductions.*
(2) *A regular torsion free $H_{1,2}$-structure with full holonomy is determined by three parameters. Thus, $H_{1,2}$-connections do exists, and $H_{1,2}$ is therefore another exotic holonomy representation.*

Comparing this result with Theorem 0.1. it follows that the torsion free $G_{1,2}$-connections do not arise as moduli spaces of rational curves in a contact threefold. However, we have the following characterization of H_3-connections.

Theorem 0.4. *Given a torsion free $G_{1,2}$-connection on a sixfold $\mathcal{Z}$ and an imbedding $\mathcal{Y} \hookrightarrow \mathcal{Z}$ of a fourfold $\mathcal{Y}$ such that the connection on $\mathcal{Z}$ restricts to a G_3-connection on $\mathcal{Y}$, then the holonomy of this restriction is contained in H_3.*

Conversely, if $\mathcal{Y}$ is a fourfold with a regular torsion free H_3-connection, then there is a unique torsion free $G_{1,2}$-connection on some sixfold $\mathcal{Z}$ and an imbedding $\mathcal{Y} \hookrightarrow \mathcal{Z}$ such that the connection on $\mathcal{Z}$ restricts to the connection on $\mathcal{Y}$.

Regularity of an H_3-connection is a generic condition. For the exact definition, see [Br].

As an interesting consequence, we conclude that for a given regular H_3-connection on $\mathcal{Y}$, there is more than one $G_{1,2}$-connection extending the H_3-connection, but exactly one of these extensions is torsion free.

The calculations in this paper make extensive use of the representation theory of $Sl(2,\mathbb{C})$ and $Sl(2,\mathbb{C}) \times Sl(2,\mathbb{C})$, particularly an explicit version of the Clebsch-Gordan formula. For details, we refer the reader to [H] and [Br].

The *Clebsch-Gordan formula* describes the irreducible decomposition of a tensor product of irreducible $Sl(2,\mathbb{C})$-modules:

$$\mathcal{V}_m \otimes \mathcal{V}_n = \bigoplus_{p=0}^{min(m,n)} \mathcal{V}_{m+n-2p}$$

A convenient tool to compute the decomposition of polynomials into their irreducible components are the bilinear pairings

$$\langle\ ,\ \rangle_p : V_n \otimes V_m \longrightarrow V_{n+m-2p}$$

$$\langle u, v\rangle_p = \frac{1}{p!}\sum_{k=0}^{p}(-1)^k \binom{p}{k} \frac{\partial^p u}{\partial^k x\, \partial^{p-k} y} \frac{\partial^p v}{\partial^{p-k} x\, \partial^k y} \quad \text{for} \quad u \in V_n, v \in V_m.$$

It can be shown that these pairings are $Sl(2,\mathbb{C})$-equivariant and therefore are the projections onto the summands of the Clebsch-Gordan formula.

The Clebsch-Gordan formula for the irreducible representations of $Sl(2,\mathbb{C}) \times Sl(2,\mathbb{C})$ reads:

$$\mathcal{V}_{i_1,i_2} \otimes \mathcal{V}_{j_1,j_2} = \bigoplus_{p_1=0}^{min(i_1,j_1)} \bigoplus_{p_2=0}^{min(i_2,j_2)} \mathcal{V}_{i_1+j_1-2p_1,\ i_2+j_2-2p_2}$$

On these spaces, we define the pairings

$$\langle\ ,\ \rangle_{p_1,p_2} : V_{i_1,i_2} \otimes V_{j_1,j_2} \longrightarrow V_{i_1+j_1-2p_1,\ i_2+j_2-2p_2}$$

defined by

$$\langle u_{i_1} \otimes v_{i_2}, u_{j_1} \otimes v_{j_2} \rangle_{p_1,p_2} := (\langle u_{i_1}, u_{j_1} \rangle_{p_1}) \otimes (\langle v_{i_2}, v_{j_2} \rangle_{p_2})$$

with the pairing $\langle\ ,\ \rangle_{p_k}$ from above.

Again, it can be shown that these pairings are $Sl(2,\mathbb{C}) \times Sl(2,\mathbb{C})$-equivariant and therefore are the projections onto the summands of the Clebsch-Gordan formula.

In §1, we define the notion of a G-structure and intrinsic torsion which will be important in §3. To demonstrate the usefulness of this concept we include several examples.

In §2, we cite Kodaira's Deformation Theorem [K] which states that, under certain circumstances, the moduli space $\mathcal{Z}$ of compact submanifolds of a given space $\mathcal{W}$ is itself a manifold. We construct a natural G-structure on this moduli space where G is the automorphism group of the normal bundle of an element of $\mathcal{Z}$ in $\mathcal{W}$, provided some stability condition (condition (A)) is satisfied.

If $\mathcal{Z}$ is the moduli space of *rational curves* with positive semistable normal bundle, then condition (A) is satisfied and hence we get a G-structure on $\mathcal{Z}$. In particular, if $\dim(\mathcal{W}) = 3$ we obtain $G_{1,k}$-structures for some positive integer k.

We then construct a certain class of connections on this $G_{1,k}$-structure, called *special connections*. These have the property that the submanifolds $\mathcal{Z}_p \subseteq \mathcal{Z}$ with $p \in \mathcal{W}$, consisting of all $C \in \mathcal{Z}$ which pass through p, are totally geodesic. This yields some information about the torsion of special connections.

In §3, we consider a *contact threefold* $\mathcal{W}$, and a rational contact curve C. We let $\mathcal{Y}$ be the moduli space of rational contact curves close to C and $\mathcal{Z}$ be the moduli space of *all* curves close to C. We then construct a G_{k+1}-structure on $\mathcal{Y}$ from the $G_{1,k}$-structure on $\mathcal{Z}$, and show that every connection on $\mathcal{Z}$ restricts naturally to a connection on $\mathcal{Y} \subseteq \mathcal{Z}$.

If $k = 2$, i.e. if the normal bundle of each $C \in \mathcal{Z}$ is equivalent to $\mathcal{O}(2) \oplus \mathcal{O}(2)$, then – using special connections – we show that the intrinsic torsion of $\mathcal{Z}$ is a section of a certain rank four bundle. This establishes Theorem 0.2. Also, the restriction of this connection to $\mathcal{Y}$ is *torsion free*, hence we get a new proof that the G_3-structure on $\mathcal{Y}$ is torsion free.

In §4, we set up the structure equations for a *torsion free* $G_{1,2}$-connection. It turns out that the first Bianchi identity forces the holonomy of such a connection to

lie in $H_{1,2} \subseteq G_{1,2}$, hence we consider the structure equations for torsion free $H_{1,2}$-connections instead. These equations and their derivatives are similar – albeit more complex – to the structure equations for H_3-connections studied in [Br]. In fact, methods similar to the ones used in [Br] allow us to solve the structure equations explicitly. Their moduli space is then computed and we prove Theorem 0.3.

Finally, in §5 we put together the results from §§3 and 4. First of all, we show that if $\mathcal{Z}$ is the moduli space of rational curves in a threefold $\mathcal{W}$ and if the associated $G_{1,2}$-structure on $\mathcal{Z}$ is torsion free, then $\mathcal{Z}$ must be locally symmetric. Second, we determine those torsion free $G_{1,2}$-structures on $\mathcal{Z}$ which restrict to a G_3-structure on some $\mathcal{Y} \subseteq \mathcal{Z}$. Since none of these structures are locally symmetric, Theorem 0.1. follows. We also demonstrate Theorem 0.4. using the classification of H_3-connections from [Br].

We conclude by discussing some questions which our investigation raises. Namely, we show that every $G_{1,2}$-structure whose torsion is a section of the bundle $T \subseteq \Lambda^2 T^*\mathcal{Z} \otimes T\mathcal{Z}$ from Theorem 0.2. is locally equivalent to the moduli space of rational curves in a fivefold $\mathcal{P}$ which integrate a rank two Pfaffian system on $\mathcal{P}$.

For example, if $\mathcal{Z}$ is the moduli space of rational curves in a threefold $\mathcal{W}$, we can achieve this by letting $\mathcal{P} := \mathbb{P}T\mathcal{W}$ with the canonical differential system [EDS], and identifying each curve $C \subseteq \mathcal{W}$ with its canonical lift $\hat{C} \subseteq \mathcal{P}$.

Of course, there are many rank two Pfaffian systems which are not locally equivalent to this contact structure on $\mathbb{P}T\mathcal{W}$ [C]. An interesting question is:

Which rank two Pfaffian systems on a fivefold $\mathcal{P}$ yield torsion free $G_{1,2}$-connections?

Since the moduli space of torsion free $G_{1,2}$-connections is only three dimensional by Theorem 0.3., those systems must be very special. The answer to this question will also shed some light onto the significance of H_3-connections. We shall pursue this analysis in a sequel of the present paper.

§1 G-structures and intrinsic torsion.

Let M^n be a (real or complex) manifold of dimension n. Let $\pi : \mathfrak{F} \to M$ be the *coframe bundle* of M, i.e. each $u \in \mathfrak{F}$ is a linear isomorphism $u : T_{\pi(u)}M \xrightarrow{\sim} \mathcal{V}$, where $\mathcal{V}$ is a fixed n-dimensional (real or complex) vector space. Then $\mathfrak{F}$ is naturally a principal right $Gl(\mathcal{V})$-bundle over M, where the right action $R_g : \mathfrak{F} \to \mathfrak{F}$ is defined by $R_g(u) = g^{-1} \circ u$. The *tautological* 1-*form* θ on $\mathfrak{F}$ with values in $\mathcal{V}$ is defined by letting $\theta(\xi) = u(\pi_*(\xi))$ for $\xi \in T_u\mathfrak{F}$. For θ, we have the $Gl(\mathcal{V})$-equivariance $R_g^*(\theta) = g^{-1}\theta$.

Let $G \subseteq Gl(\mathcal{V})$ be a closed Lie subgroup and let $\mathfrak{g} \subseteq \mathfrak{gl}(\mathcal{V})$ be the Lie algebra of G. A G-structure on M is, by definition, a G-subbundle $F \subseteq \mathfrak{F}$. For any G-structure, we will denote the restrictions of π and θ to F by the same letters. Given $A \in \mathfrak{g}$ we define the vector field A_* on F by

$$(A_*)_u = \frac{d}{dt}\left(R_{exp(tA)}(u)\right)|_{t=0}.$$

The vector fields A_* are called the *fundamental vertical vector fields* on F. It is evident that $\pi_*(A_*) = 0$ and thus $\theta(A_*) = 0$ for all $A \in \mathfrak{g}$; in fact, $\{A_*|A \in \mathfrak{g}\} = \ker(\pi_*)$. Moreover, for $A, B \in \mathfrak{g}$ it is well-known that $[A_*, B_*] = [A, B]_*$.

Let $x \in M$ and $u \in \pi^{-1}(x)$. The Lie algebra $\mathfrak{g}_x := u^{-1}\mathfrak{g}u \subseteq \mathfrak{gl}(T_xM)$ is independent of the choice of u, and the union $\mathfrak{g}_F := \bigcup_x \mathfrak{g}_x$ is a vector subbundle of $T^*M \otimes TM$.

Now we shall consider the *first Spencer map* $Sp : \mathcal{V}^* \otimes \mathfrak{gl}(\mathcal{V}) \to \Lambda^2\mathcal{V}^* \otimes \mathcal{V}$ which is defined by skew-symmetrization of the first two factors of $\mathcal{V}^* \otimes \mathfrak{gl}(\mathcal{V}) \cong \mathcal{V}^* \otimes \mathcal{V}^* \otimes \mathcal{V}$. Since $\mathfrak{g} \subseteq \mathfrak{gl}(\mathcal{V})$ we may consider the restriction of Sp to $\mathcal{V}^* \otimes \mathfrak{g}$, and we define $\mathfrak{g}^{(1)}$ and $H^{0,2}(\mathfrak{g})$ by requiring that the following sequence be exact:

$$0 \longrightarrow \mathfrak{g}^{(1)} \longrightarrow \mathcal{V}^* \otimes \mathfrak{g} \xrightarrow{Sp} \Lambda^2\mathcal{V}^* \otimes \mathcal{V} \xrightarrow{pr} H^{0,2}(\mathfrak{g}) \longrightarrow 0. \tag{1-1}$$

In the same way, we can define vector bundles $\mathfrak{g}_F^{(1)}$ and $H_F^{0,2}$ over M via the exact sequence

$$0 \longrightarrow \mathfrak{g}_F^{(1)} \longrightarrow T^*M \otimes \mathfrak{g}_F \xrightarrow{Sp} \Lambda^2T^*M \otimes TM \xrightarrow{pr} H_F^{0,2} \longrightarrow 0. \tag{1-2}$$

From now on, we will denote points in M by x and points in F by u. Moreover, ξ, ξ' denote tangent vectors on F and we let $\underline{\xi}_u = \pi_*(\xi_u)$, $\underline{\xi}'_u = \pi_*(\xi'_u)$ etc.

A *connection* on F is a $\mathfrak{g}$-valued 1-form ω on F satisfying the conditions

$$\begin{array}{ll} \omega(A_*) = A & \text{for all } A \in \mathfrak{g}, \text{ and} \\ R_g^*(\omega) = g^{-1}\omega g & \text{for all } g \in G. \end{array} \tag{1-3}$$

Given a connection ω, its *torsion* Θ is the $\mathcal{V}$-valued 2-form given by

$$\Theta = d\theta + \omega \wedge \theta. \tag{1-4}$$

From (1-3) and (1-4) it follows that there is a section Tor of $\Lambda^2T^*M \otimes TM$ satisfying

$$\Theta(\xi_u, \xi'_u) = u\ (Tor(\underline{\xi}_u, \underline{\xi}'_u)) \quad \text{for all } \xi_u, \xi'_u \in T_uF \text{ and all } u \in F. \tag{1-5}$$

The connection ω is called *torsion free* if $\Theta = 0$.

Now let ω' be another connection on F, and let Θ' and Tor' represent its torsion. From (1-3) it follows that there is a section α of the bundle $T^*M \otimes \mathfrak{g}_F$ such that

$$(\omega' - \omega)(\xi_u) = u\ \alpha(\underline{\xi}_u)\ u^{-1}. \tag{1-6}$$

From (1-4) - (1-6) we obtain for the torsion

$$(\Theta' - \Theta)(\xi_u, \xi'_u) = u\ \left(\alpha(\underline{\xi}_u) \cdot \underline{\xi}'_u - \alpha(\underline{\xi}'_u) \cdot \underline{\xi}_u\right),$$

and hence,

$$\begin{aligned} (Tor' - Tor)(\underline{\xi}, \underline{\xi}') &= \alpha(\underline{\xi}) \cdot \underline{\xi}' - \alpha(\underline{\xi}') \cdot \underline{\xi} \\ &= Sp(\alpha)(\underline{\xi}, \underline{\xi}') \qquad \text{for all } \underline{\xi}, \underline{\xi}' \in T_xM. \end{aligned}$$

Thus, we conclude that

$$Tor' = Tor + Sp(\alpha). \tag{1-7}$$

This implies that the section $\tau := pr(Tor)$ of the bundle $H_F^{0,2}$ is independent of the choice of ω and therefore *only depends on the G-structure F*.

Definition 1.1. Let $\pi : F \to M$ be a G-structure.

(1) The vector bundle $H_F^{0,2}$ is called the *intrinsic torsion bundle of* F.
(2) The section τ of $H_F^{0,2}$ defined above is called the *intrinsic torsion of* F.
(3) F is called *torsion free* if its intrinsic torsion $\tau = 0$.

The following Proposition is then immediate from (1-7).

Proposition 1.2. *Let $\pi : F \to M$ be a G-structure and let τ be its intrinsic torsion.*

(1) *If σ is any section of $\Lambda^2 T^*M \otimes TM$ such that $pr(\sigma) = \tau$ then there is a connection on F whose torsion section Tor equals σ.*
(2) *There is a torsion free connection on F if and only if F is torsion free.*
(3) *If F is torsion free then there is a one-to-one correspondence between torsion free connections on F and sections of $\mathfrak{g}_F^{(1)}$. In particular, if $\mathfrak{g}^{(1)} = 0$ then the torsion free connection on F is unique.*

We will give some examples for this concept.

Examples 1.3.

(1) Let $G = O(p,q) \subseteq Gl(\mathcal{V})$ with $\mathcal{V} = \mathbb{R}^n$ and $n = p+q$. A G-structure on M^n is equivalent to a pseudo-Riemannian metric on M of signature (p,q). One can show that $Sp : \mathcal{V}^* \otimes \mathfrak{o}(p,q) \to \Lambda^2\mathcal{V}^* \otimes \mathcal{V}$ is an *isomorphism*. Thus, $\mathfrak{g}^{(1)} = 0$ and $H^{0,2}(\mathfrak{g}) = 0$. Then Proposition 1.2. implies that there is a *unique torsion free connection* on such a G-structure. Of course, this reproves precisely the existence and uniqueness of the Levi-Civita connection of a (pseudo-)Riemannian metric. [KN]
(2) Suppose $n = 2m$ and let $G = Gl(m,\mathbb{C}) \subseteq Gl(n,\mathbb{R})$. A G-structure on M^n is equivalent to an almost complex structure on M. Then $H^{0,2}(\mathfrak{gl}(m,\mathbb{C})) = \{\phi \in \Lambda^2(\mathbb{C}^n)^* \otimes_{\mathbb{R}} \mathbb{C}^n \mid \phi(ix,y) = -i\phi(x,y)\}$. Moreover, the intrinsic torsion is given by the *Nijenhuis tensor*. It is well known that the vanishing of this tensor, i.e. the torsion freeness of the G-structure, is equivalent to the integrability of the almost complex structure. [KN]
(3) Suppose $n = 2m$ and let $G = Sp(m) \subseteq Gl(n,\mathbb{R})$. A G-structure on M^n is equivalent to a non-degenerate 2-form ω on M, i.e. $\omega^m \neq 0$. One can show that $H^{0,2}(\mathfrak{sp}(m)) = \Lambda^3\mathbb{R}^n$ and that the intrinsic torsion is represented by the 3-form $d\omega$. Thus, the G-structure is torsion free if anf only if ω is a symplectic form.

From these examples it should become evident that for many naturally arising G-structures the vanishing of the intrinsic torsion implies, in some sense, the "most natural integrability condition" of the underlying geometric structure.

§2 *G*-structures on moduli spaces of compact submanifolds.

Let $\mathcal{W}$ be a complex manifold of (complex) dimension $d + r$.

Definition 2.1. By an *analytic family of compact submanifolds of dimension d of* $\mathcal{W}$ we shall mean a pair $(\mathcal{N}, \mathcal{Z})$ of a complex manifold $\mathcal{Z}$ and a complex analytic submanifold $\mathcal{N}$ of $\mathcal{W} \times \mathcal{Z}$ of codimension r with the property that for each $t \in \mathcal{Z}$, the intersection $\mathcal{N} \cap (\mathcal{W} \times t)$ is a compact connected submanifold of $\mathcal{W} \times t$ of dimension d.

We call $\mathcal{Z}$ the *moduli space* of the family $(\mathcal{N}, \mathcal{Z})$. The canonical projections of $\mathcal{N}$ onto $\mathcal{W}$ and $\mathcal{Z}$ will be denoted by pr_1 and pr_2 respectively.

$$\begin{tikzcd} & \mathcal{N} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathcal{W} & & \mathcal{Z} \end{tikzcd}$$

For each point $t \in \mathcal{Z}$, we set

$$C_t \times t = \mathcal{N} \cap (\mathcal{W} \times t).$$

We may identify $C_t \times t$ with C_t and consider $\mathcal{N}$ as a *family consisting of compact submanifolds C_t, $t \in \mathcal{Z}$, of $\mathcal{W}$*.

From now on, we shall use the notational convention $H^i(E) := H^i(C, \mathcal{O}(E))$ for a vector bundle $E \to C$.

Let $N_t \to C_t$ be the normal bundle of C_t in $\mathcal{W}$. There is a natural imbedding $\eta_t : T_t\mathcal{Z} \hookrightarrow H^0(N_t)$ [K] and we shall use η_t to regard $T_t\mathcal{Z}$ as a subspace of $H^0(N_t)$.

Definition 2.2. An analytic family $(\mathcal{N}, \mathcal{Z})$ is called *complete at $t \in \mathcal{Z}$* if η_t is an isomorphism. $(\mathcal{N}, \mathcal{Z})$ is called *complete* if it is complete at t for all $t \in \mathcal{Z}$.

We now state one of the most famous classical Theorems of the subject:

Kodaira's Deformation Theorem [K]**.** *Let $C \subseteq \mathcal{W}$ be a compact submanifold of $\mathcal{W}$ of dimension d. Let $N \to C$ be the normal bundle of C in $\mathcal{W}$. If $H^1(N) = 0$ then there exists a complete analytic family $(\mathcal{N}, \mathcal{Z})$ such that $C = C_{t_0}$ for some $t_0 \in \mathcal{Z}$.*

Let $E \to C$ be a holomorphic vector bundle and denote the group of equivalences of E with itself by $Aut(E)$. Since each $\phi \in Aut(E)$ induces an isomorphism $\hat{\phi} : \mathcal{O}(E) \to \mathcal{O}(E)$, we obtain a natural representation $\alpha_* : Aut(E) \to Gl(H^*(E))$.

Definition 2.3. An analytic family $(\mathcal{N}, \mathcal{Z})$ is said to *satisfy condition* (A) if

(i) for any $t_1, t_2 \in \mathcal{Z}$ the normal bundles $N_{t_i} \to C_{t_i}$, $i = 1, 2$, are equivalent, and

(ii) the representation $\alpha_0 : Aut(N_t) \to Gl(H^0(N_t))$ is faithful and has closed image for all $t \in \mathcal{Z}$.

Consider now a complete analytic family $(\mathcal{N}, \mathcal{Z})$ satisfying condition (A). Let $E \to C$ be a vetor bundle which is equivalent to the normal bundles $N_t \to C_t$ for all $t \in \mathcal{Z}$, and let $G := Aut(E)$. Let $\mathcal{V} := H^0(E)$, and let

$$\pi : \mathfrak{F} \to \mathcal{Z}$$

be the $\mathcal{V}$-coframe bundle of $\mathcal{Z}$. Now consider the principal bundle $\pi : F \to \mathcal{Z}$ with

$$F := \left\{ \imath : \begin{array}{ccc} E & \longrightarrow & N_t \\ \downarrow & & \downarrow \\ C & \longrightarrow & C_t \end{array} \middle| \begin{array}{c} t \in \mathcal{Z}, \\ \imath \text{ a bundle equivalence.} \end{array} \right\}. \tag{2-1}$$

We can define a bundle imbedding $\zeta : F \hookrightarrow \mathfrak{F}$ (and thereby justify the double use of the symbol π) as follows: given $\imath \in F$, there is an induced isomorphism

$\hat{\imath} : \mathcal{V} \to H^0(N_t)$. Then $\eta_t \circ \hat{\imath} : \mathcal{V} \to T_t\mathcal{Z}$ is also a linear isomorphism, thus $(\eta_t \circ \hat{\imath})^{-1}$ is a point in $\mathfrak{F}$. From condition (A) it follows that the definition $\zeta(\imath) := (\eta_t \circ \hat{\imath})^{-1}$ is indeed one-to-one, and moreover, the image $\zeta(F)$ is a G-structure on $\mathcal{Z}$. Identifying F with $\zeta(F)$, we regard F as a G-structure on $\mathcal{Z}$, where the principal G-action on F is given by $R_g(\imath) := \imath \circ g$.

If we set $\underline{\theta} := \zeta^*(\theta)$, where θ is the tautological form on $\mathfrak{F}$, then

$$\underline{\theta}(\xi_\imath) = \zeta(\imath)((\pi \circ \zeta)_*(\xi_\imath)). \tag{2-2}$$

By a slight abuse of language, we shall call $\underline{\theta}$ the *tautological form on* F.

Thus, for a complete analytic family $(\mathcal{N}, \mathcal{Z})$ *satisfying condition* (A)*, we have constructed an induced* G*-structure on the moduli space* $\mathcal{Z}$*.*

For the remainder of this section, $(\mathcal{N}, \mathcal{Z})$ will stand for a complete analytic family of *rational curves* satisfying condition (A), i.e. we assume that $d = 1$ and $C_t \cong \mathbb{P}^1$ for all $t \in \mathcal{Z}$.

It is well-known that every k-dimensional vector bundle $E \to \mathbb{P}^1$ satisfies $\mathcal{O}(E) \cong \mathcal{O}(m_1) \oplus \ldots \oplus \mathcal{O}(m_k)$ for some integers m_i, $i = 1, \ldots, k$. Moreover, it is not hard to show that E satisfies condition (ii) in Definition 2.3. if and only if $m_i \geq 0$ for all i and $m_1 + \ldots + m_k > 0$. In this case, the automorphism group decomposes as

$$G \cong \tilde{G} \times Sl(2, \mathbb{C}) \quad \text{with} \quad \tilde{G} \cong Gl(n_1, \mathbb{C}) \times \ldots \times Gl(n_l, \mathbb{C}),$$

where the n_i's are the multiplicities of the m_i's. [GH]

An interesting question is to determine the intrinsic torsion of such a G-structure or at least to understand its *vanishing*. To do this, we will construct connections on F and make some statements about their torsion.

Let

$$\rho : Sl(2, \mathbb{C}) \times \mathbb{P}^1 \to \mathbb{P}^1$$

denote the action of $Sl(2, \mathbb{C})$ on $\mathbb{P}^1$ by Möbius transformations. Let us fix once and for all the reference point

$$x_0 := [0 : 1] \in \mathbb{P}^1.$$

Consider $(\mathcal{N}, \mathcal{Z})$ as before. Let

$$P := \left\{ \underline{\imath} : \mathbb{P}^1 \to C_t \,\middle|\, t \in \mathcal{Z},\ \underline{\imath} \text{ a biholomorphism} \right\}$$

be the *parameter space of* $\mathcal{Z}$. Then the obvious projection

$$\pi_{P,\mathcal{Z}} : P \to \mathcal{Z}$$

is a principal $Sl(2, \mathbb{C})$-bundle, where the principal action is defined by $R_g(\underline{\imath}) = \underline{\imath} \circ \rho(g)$. There is another projection

$$\pi_{F,P} : F \to P$$

which maps a bundle isomorphism $\imath : E \to N_t$ to the underlying biholomorphism $\underline{\imath} : \mathbb{P}^1 \to C_t$. $\pi_{F,P}$ yields another principal bundle with structure group $\tilde{G} = G/Sl(2, \mathbb{C})$. Finally, there is a projection

$$\pi \ : P \to \mathcal{N}$$

which projects $\underline{\imath} : \mathbb{P}^1 \to C_t$ to $(\underline{\imath}(x_0), t) \in \mathcal{N}$. This projection yields a principal bundle whose structure group is the stabilizer $G_{x_0} \subseteq G$.

Summarizing, we have the following commutative diagram:

$$
\begin{tikzcd}
F \arrow[d, "\pi_{F,P}"'] & & \\
P \arrow[dd, "\pi_{P,\mathcal{Z}}"'] \arrow[dr, "\pi_{P,\mathcal{N}}"] & & \\
& \mathcal{N} \arrow[dl, "pr_2"'] \arrow[dr, "pr_1"] & \\
\mathcal{Z} & & \mathcal{W}
\end{tikzcd}
$$

Now let us fix $t_0 \in \mathcal{Z}$ and let $C := C_{t_0}$. If we denote the tangent and the normal bundle of C by τ_C and N_C respectively then we have the exact sequence

$$0 \to \tau_C \to T\mathcal{W}|_C \to N_C \to 0, \tag{2-3}$$

where $T\mathcal{W}$ denotes the holomorphic tangent bundle of $\mathcal{W}$.

It is well known that this sequence splits. Also, from (2-3) we get the exact sequence

$$0 \to Hom(N_C, \tau_C) \to Hom(T\mathcal{W}|_C, \tau_C) \to Hom(\tau_C, \tau_C) \to 0, \tag{2-4}$$

which in turn induces the exact sequence

$$0 \longrightarrow H^0(Hom(N_C, \tau_C)) \longrightarrow H^0(Hom(T\mathcal{W}|_C, \tau_C)) \xrightarrow{\pi^*} H^0(Hom(\tau_C, \tau_C)) \tag{2-5}$$

Let

$$S_C := (\pi^*)^{-1}(id_{\tau_C}) \subseteq H^0(Hom(T\mathcal{W}|_C, \tau_C)),$$

where id_{τ_C} is regarded as an element of $H^0(Hom(\tau_C, \tau_C))$. Even though π^* need not be surjective in general, S_C is non-empty; namely, S_C consists of all splitting maps of the exact sequence (2-3). Therefore, S_C is an affine subspace of $H^0(Hom(T\mathcal{W}|_C, \tau_C))$ whose dimension equals that of $H^0(Hom(N_C, \tau_C))$.

Condition (A) implies that the exact sequences (2-3) - (2-5) are independent of the choice of $t_0 \in \mathcal{Z}$, hence so is the dimension of S_C. In fact, the union

$$S := \bigcup_{t \in \mathcal{Z}} S_{C_t}$$

forms an affine bundle over $\mathcal{Z}$, called the *split-bundle of $\mathcal{Z}$*.

Lemma 2.4. *Given a local section $\sigma : \mathcal{U} \to S$, $t \mapsto \sigma_t$ of the split-bundle S, let $P_{\mathcal{U}} := \pi_{P,\mathcal{Z}}^{-1}(\mathcal{U})$. There is a unique holomorphic connection $\hat{\sigma}$ on $\pi_{P,\mathcal{Z}} : P_{\mathcal{U}} \to \mathcal{U}$ such that*

$$\rho_*(\hat{\sigma}(\xi), 0_{x_0}) = \underline{\imath}_*^{-1}(\sigma_t(\underline{\xi})) \tag{2-6}$$

for all $\xi \in T_{\underline{\imath}}P$, where $t = \pi_{P,\mathcal{Z}}(\underline{\imath})$ and $\underline{\xi} = (pr_1 \circ \pi_{P,\mathcal{N}})_*(\xi)$.

Proof. First of all, note that equation (2-6) is well defined: $\hat{\sigma}(\xi) \in \mathfrak{sl}(2, \mathbb{C})$, and therefore both sides are contained in $T_{x_0}\mathbb{P}^1$.

Let $\underline{\sigma}(\xi)$ be the right hand side of (2-6). Then $\underline{\sigma}$ is a holomorphic 1-form on $P_{\mathcal{U}}$ with values in $T_{x_0}\mathbb{P}^1$. Moreover, it is easy to see that $\rho_*(A, 0_{x_0}) = \underline{\sigma}(A_*)$ for all $A \in \mathfrak{sl}(2,\mathbb{C})$, where A_* denotes the fundamental vector field corresponding to A.

We define a basis $\{A_1, A_2, A_3\}$ of $\mathfrak{sl}(2,\mathbb{C})$ by the equation

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = aA_1 + bA_2 + cA_3.$$

Clearly, $\rho_*(A_i, 0_{x_0}) = 0$ for $i = 1, 2$. We define the complex-valued 1-form $\hat{\sigma}_3$ by the equation $\hat{\sigma}_3(\xi)\ \rho_*(A_3, 0_{x_0}) = \underline{\sigma}(\xi)$, and let $\hat{\sigma}_1 := \mathfrak{L}_{A_2}(\hat{\sigma}_3)$ and $\hat{\sigma}_2 := \frac{1}{2}\mathfrak{L}_{A_2}(\hat{\sigma}_1)$, where $\mathfrak{L}$ denotes the Lie derivative.

It is left to the reader to verify that the $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form

$$\hat{\sigma} := \sum_i \hat{\sigma}_i A_i$$

defines a connection with the desired property and that this choice is unique. $\square$

Geometrically, the interpretation of the connection $\hat{\sigma}$ is the following. Suppose we have a local section $\sigma : \mathcal{U} \to S$ and a curve $\gamma : I \to \mathcal{U} \subseteq \mathcal{Z}$ for some open set $I \subseteq \mathbb{C}$, $0 \in I$. Then a horizontal lift $\overline{\gamma} : I \to P$ of γ can be regarded as a map $\Gamma : I \times \mathbb{P}^1 \to \mathcal{W}$ such that $\Gamma(t, \_)$ parametrizes $\gamma(t)$.

Given a parametrization $\underline{\imath} : \mathbb{P}^1 \to \gamma(0)$, we then define Γ uniquely by requiring that

(1) $\Gamma(0, x) = \underline{\imath}(x)$ for all $x \in \mathbb{P}^1$, and
(2) $\sigma_t(\frac{\partial}{\partial t}\Gamma(t, x)) = 0$ for all $t \in I$ and all $x \in \mathbb{P}^1$.

It is then easy to verify that the Γ thus determined is the horizontal lift of γ w.r.t. the connection $\hat{\sigma}$.

Definition 2.5. A holomorphic connection ω on $\pi : F \to \mathcal{Z}$ is called *special* if there exists a holomorphic section σ of the split-bundle $S \to \mathcal{Z}$, and a $\tilde{\mathfrak{g}}$-valued 1-form $\tilde{\omega}$ on F such that

$$\omega = \tilde{\omega} + \hat{\sigma}$$

with $\hat{\sigma}$ as in Lemma 2.4. Here, we use the decomposition $\mathfrak{g} \cong \tilde{\mathfrak{g}} \oplus \mathfrak{sl}(2,\mathbb{C})$.

Proposition 2.6. *Let $(\mathcal{N}, \mathcal{Z})$ and $\pi : F \to \mathcal{Z}$ be as before. Every $t_0 \in \mathcal{Z}$ has a neighborhood $\mathcal{U} \subseteq \mathcal{Z}$ such that the restricted bundle $\pi : F_{\mathcal{U}} \to \mathcal{U}$ with $F_{\mathcal{U}} := \pi^{-1}(\mathcal{U})$ admits a special connection.*

Proof. The proof is almost obvious: choose $\mathcal{U}$ sufficiently small such that $\pi : F_{\mathcal{U}} \to \mathcal{U}$ admits a holomorphic connection $\underline{\omega} = \tilde{\omega} + \phi$ where $\tilde{\omega}$ and ϕ are holomorphic 1-forms with values in $\tilde{\mathfrak{g}}$ and $\mathfrak{sl}(2,\mathbb{C})$ respectively. After shrinking $\mathcal{U}$ we may also assume that the split-bundle S admits a holomorphic section σ over $\mathcal{U}$. Then the form $\omega := \tilde{\omega} + \hat{\sigma}$ is a special connection. $\square$

An important characterization of special connections comes from the following

Proposition 2.7. *For every $p \in \mathcal{W}$, let $\mathcal{Z}_p := \{t \in \mathcal{Z} | p \in C_t\}$. If $\mathcal{Z}_p \neq \emptyset$ then $\mathcal{Z}_p$ is a smooth submanifold of $\mathcal{Z}$ with $\operatorname{codim}(\mathcal{Z}_p) = \dim(\mathcal{W}) - 1$. The tangent space of $\mathcal{Z}_p$ at $t \in \mathcal{Z}_p$ is $H^0(C_t, \mathcal{O}(N_t) - p) \subseteq H^0(N_t) \cong T_t\mathcal{Z}$. Moreover, $\mathcal{Z}_p$ is totally geodesic w.r.t. any special connection ω.*

Proof. The proof of the first two parts is left to the reader.

To show that $\mathcal{Z}_p$ is totally geodesic, let $t_0 \in \mathcal{Z}_p$ and pick a biholomorphism $\underline{\imath}_0 : \mathbb{P}^1 \to C_{t_0}$ such that $\underline{\imath}_0(x_0) = p$. Then $\underline{\imath}_0 \in \pi^{-1}_{P,\mathcal{Z}}(t_0)$. We also pick a bundle isomorphism $\imath_0 : E \to N_{t_0}$ such that $\imath_0 \in \pi^{-1}_{F,P}(\underline{\imath}_0)$. Here, $E \to \mathbb{P}^1$ is a vector bundle which is isomorphic to $\mathcal{O}(N_t) \to C_t$ for all $t \in \mathcal{Z}$.

Let $\omega = \tilde{\omega}+\hat{\sigma}$ be a special connection on F where σ is a section of the split-bundle S. Let $I \subseteq \mathbb{C}$ be an open neighborhood of 0, and consider a geodesic $\gamma : I \to \mathcal{Z}$ with $\gamma(0) = t_0$ and $\gamma'(0) \in T_{t_0}\mathcal{Z}_p$. Let $\underline{\tilde{\gamma}} : I \to P$ and $\tilde{\gamma} : I \to F$ be the horizontal lifts of γ to P and F respectively with $\underline{\tilde{\gamma}}(0) = \underline{\imath}_0$ and $\tilde{\gamma}(0) = \imath_0$.

Define $\Gamma : I \times \mathbb{P}^1 \to \mathcal{W}$ by $\Gamma(z, x) := \underline{\tilde{\gamma}}(z)(x)$. Since $\tilde{\gamma}$ is horizontal and thus, in particular, $\hat{\sigma}(\underline{\tilde{\gamma}}'(z)) = 0$ for all z, it follows that

$$\sigma_{\gamma(z)} \left(\tfrac{\partial}{\partial z}\Gamma(z, x_0)\right) = 0 \quad \text{for all } z \in I. \tag{2-7}$$

On the other hand, since γ is a geodesic, we conclude from (2-2) that $\zeta(\tilde{\gamma}(z))(\gamma'(z)) \in H^0(E)$ is independent of z and thus vanishes at x_0 for all z. It follows that

$$\tfrac{\partial}{\partial z}\Gamma(z, x_0) \quad \text{is } \textit{tangent to } C_{\gamma(z)} \text{ for all } z \in I. \tag{2-8}$$

But (2-7) and (2-8) together imply that

$$\tfrac{\partial}{\partial z}\Gamma(z, x_0) = 0 \quad \text{for all } z \in I,$$

and thus $\Gamma(z, x_0) = p$ for all $z \in I$. But this means $\gamma(z) \in \mathcal{Z}_p$ for all z and this completes the proof. □

This Proposition yields immediately

Corollary 2.8. *If ω is a special connection on $\mathcal{Z}$ and $Tor : \Lambda^2 H^0(N_t) \to H^0(N_t)$ is its torsion then*

$$Tor(\Lambda^2 H^0(C_t, \mathcal{O}(N_t) - p)) \subseteq H^0(C_t, \mathcal{O}(N_t) - p)$$

for all $p \in C_t$. □

Since torsion is a *local* concept, Proposition 2.6. together with Corollary 2.8 will allow us to make some assumptions about the intrinsic torsion of F. This will be applied in the following sections.

We shall also need one further property of special connections. Its proof is immediate from (2-2) and (2-6).

Proposition 2.9. *Let $\underline{\mathcal{V}} := H^0(\mathbb{P}^1, \mathcal{O}(k) \oplus \mathcal{O}(k) - x_0)$ be the space of global sections of $\mathcal{O}(k) \oplus \mathcal{O}(k)$ which vanish at x_0 and let $\mathfrak{g}' := \tilde{\mathfrak{g}} \oplus \mathfrak{h}_{x_0} \subseteq \mathfrak{g}$ where $\mathfrak{h}_{x_0} \subseteq \mathfrak{sl}(2,\mathbb{C})$ is the infinitesimal stabilizer of x_0 under ρ. Consider the projection $pr_1 \circ \pi_{F,\mathcal{N}} : F \to \mathcal{W}$. If ω is a special connection on F then*

$$\ker(pr_1 \circ \pi_{F,\mathcal{N}})_* = \{\xi \in TF \mid (\theta + \omega)(\xi) \in \underline{\mathcal{V}} \oplus \mathfrak{g}'\}. \quad \square$$

§3 Moduli spaces of rational contact curves.

In this entire section, we shall assume that $\dim(\mathcal{W}) = 3$ and that $\mathcal{W}$ carries a holomorphic contact structure, i.e. a holomorphic line bundle $L \subseteq T^*\mathcal{W}$ with the property that for every non-vanishing local section κ in L, the local 3-form $\kappa \wedge d\kappa$ does not vanish anywhere.

By a standard notational ambiguity we will denote by $\mathcal{O}(n)$ both the (unique) line bundle of degree n over $\mathbb{P}^1$ and the sheaf of germs of holomorphic sections of this line bundle.

Let us first of all cite the following

Proposition 3.1. [Br] *Let $\mathcal{W}$ denote a complex contact 3-fold with contact line bundle $L \subseteq T^*\mathcal{W}$. Let $C \subseteq \mathcal{W}$ be an imbedded rational contact curve, and suppose that $L|_C \cong \mathcal{O}(-k-1)$ for some integer $k \geq 0$. Then*

(1) *$N_C \cong \mathcal{O}(k) \oplus \mathcal{O}(k)$, where N_C denotes the normal bundle of C in $\mathcal{W}$,*
(2) *the moduli space $\mathcal{Z}$ of imbedded rational curves is smooth and of complex dimension $2k+2$ near C, and*
(3) *the subspace $\mathcal{Y} \subseteq \mathcal{Z}$ of rational contact curves in $\mathcal{W}$ is a smooth analytic submanifold of $\mathcal{Z}$ of dimension $k+2$.*

For the remainder of this section we shall assume that $k > 0$. It follows that $\mathcal{Z}$ is a *complete analytic family of rational curves satisfying condition* (A).

Let $E := \mathcal{O}(k) \oplus \mathcal{O}(k)$. Then

$$G := Aut(E) = Gl(2,\mathbb{C}) \times Sl(2,\mathbb{C}),$$

where the first factor $Gl(2,\mathbb{C})$ consists of those automorphisms which fix the base space $\mathbb{P}^1$, and the second factor $Sl(2,\mathbb{C})$ consists of automorphisms which are induced by Möbius transformations of $\mathbb{P}^1$. As an $Aut(E)$-module, $T_C\mathcal{Z} \cong H^0(\mathcal{O}(k) \oplus \mathcal{O}(k)) \cong \mathcal{V}_{1,k}$.

Let $L^\perp \subseteq T\mathcal{W}$ be the 2-plane bundle anihilated by the sections of L. For local sections ξ, ξ' and κ of $L^\perp$ and L respectively, the pairing $(\xi \wedge \xi', \kappa) \mapsto d\kappa(\xi, \xi')$ is easily seen to be tensorial and non-degenerate, hence induces a bundle isomorphism $\Lambda^2(L^\perp) \xrightarrow{\cong} L^*$. Also, we have the canonical short exact sequence

$$0 \longrightarrow L^\perp \longrightarrow T\mathcal{W} \longrightarrow L^* \longrightarrow 0, \tag{3-1}$$

where $T\mathcal{W}$ is the holomorphic tangent bundle of $\mathcal{W}$.

Now let $C \in \mathcal{Y} \subseteq \mathcal{Z}$. Since C is a contact curve, we have an inclusion $0 \to \tau \to L^\perp_{|C}$, where τ is the tangent bundle of C, and from there it follows that

$$\tau \otimes (L^\perp_{|C}/\tau) \cong \Lambda^2(L^\perp_{|C}) \cong L^*_{|C} \cong \mathcal{O}(k+1).$$

Thus, since $\tau \cong \mathcal{O}(2)$, we must have $L^\perp_{|C}/\tau \cong \mathcal{O}(k-1)$. From (3-1) we also have the short exact sequence

$$0 \longrightarrow L^\perp_{|C}/\tau \longrightarrow N_C \longrightarrow L^*_{|C} \longrightarrow 0, \tag{3-2}$$

where N_C denotes the normal bundle of C in $\mathcal{W}$.

Since $H^1(L^\perp_{|C}/\tau) \cong H^1(\mathcal{O}(k-1)) = 0$, (3-2) induces the short exact sequence

$$0 \longrightarrow H^0(L^\perp_{|C}/\tau) \xrightarrow{\imath} H^0(N_C) \xrightarrow{pr} H^0(L^*_{|C}) \longrightarrow 0.$$

Lemma 3.2. *Let $C \in \mathcal{Y}$ and $\xi \in T_C\mathcal{Y} \subseteq T_C\mathcal{Z} \cong H^0(N_C)$. If $pr(\xi) \in H^0(L^*_{|C})$ vanishes at $p \in C$ of order at least two, then ξ – regarded as a section of N_C – vanishes at p.*

Proof. Given $\xi \in T_C\mathcal{Y}$ as above, we pick a holomorphic curve $\gamma : I \to \mathcal{Y}$ with $\gamma(0) = C$ and $\gamma'(0) = \xi$ where $I \subseteq \mathbb{C}$ is an open neighborhood of 0. Let $\Gamma : I \times \mathbb{P}^1 \to \mathcal{W}$ be a holomorphic map such that $\Gamma(t, \_)$ is a parametrization of $\gamma(t)$ for all t. We may assume $\Gamma(0, x_0) = p$ with $x_0 := [0 : 1] \in \mathbb{P}^1$.

First, suppose that Γ is a local biholomorphism from a neighborhood of $(0, x_0)$ to $U \subseteq \mathcal{W}$. Then the holomorphic vector fields

$$X := \frac{\partial\Gamma}{\partial t}(t, x) \quad \text{and} \quad Y := \frac{\partial\Gamma}{\partial x}(t, x)$$

are well defined on U.

Let κ be a local contact form on U. Then we have

$$d\kappa(X_p, Y_p) = X_p(\kappa(Y)) - Y_p(\kappa(X)) - \kappa([X, Y]_p).$$

But all three terms on the right hand side vanish: the first one vanishes because Y is tangent to the contact curves $\gamma(t)$, thus $\kappa(Y) \equiv 0$. The second one vanishes because – by hypothesis – the function $\kappa(X) : (U \cap C) \to \mathbb{C}$ vanishes of order two at p. Finally, $[X, Y] = 0$ from the definition of X and Y, thus the third term vanishes as well.

The vanishing of $pr(\xi)$ at p implies that $X_p \in L^\perp_p$, hence X_p, Y_p span $L^\perp_p$. But this together with $d\kappa(X_p, Y_p) = 0$ implies that $(\kappa \wedge d\kappa)_p = 0$ which is impossible.

Therefore, Γ is *not* a local biholomorphism at $(0, x_0)$, i.e. $\frac{\partial\Gamma}{\partial t}(0, x_0)$ must be *tangent to* C. But this implies exactly that ξ vanishes at p. $\square$

Corollary 3.3. *For every $C \in \mathcal{Y}$, the restriction $pr : T_C\mathcal{Y} \to H^0(L^*_{|C})$ is an isomorphism.*

Proof. From Proposition 3.1. we know that $\dim(T_C\mathcal{Y}) = k + 2 = \dim(H^0(L^*_{|C}))$, and from Lemma 3.2. it follows that $\ker(pr) \cap T_C\mathcal{Y} = 0$. $\square$

Recall that the sequence (3-2) is equivalent to

$$0 \to \mathcal{O}(k-1) \to \mathcal{O}(k) \oplus \mathcal{O}(k) \to \mathcal{O}(k+1) \to 0.$$

It is easy to show that – up to equivalence – the maps in this exact sequence are uniquely determined. More specifically, one can show that for every $C \in \mathcal{Y}$, there are bundle isomorphisms ϕ_C, ϕ'_C and ϕ''_C such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(k-1) \arrow[r] \arrow[d, "\phi'_C"'] & \mathcal{O}(k) \oplus \mathcal{O}(k) \arrow[r] \arrow[d, "\phi_C"'] & \mathcal{O}(k+1) \arrow[r] \arrow[d, "\phi''_C"'] & 0 \\ 0 \arrow[r] & L^\perp_{|C}/\tau \arrow[r] & N_C \arrow[r] & L^*_{|C} \arrow[r] & 0 \end{tikzcd} \tag{3-3}$$

commutes, and for the induced commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{V}_{k-1} \arrow[r, "\underline{\imath}"] \arrow[d, "(\phi'_C)^*"'] & \mathcal{V}_{1,k} \arrow[r, "\underline{pr}"] \arrow[d, "\phi^*_C"'] & \mathcal{V}_{k+1} \arrow[r] \arrow[d, "(\phi''_C)^*"'] & 0 \\ 0 \arrow[r] & H^0(L^\perp_{|C}/\tau) \arrow[r, "\imath"] & H^0(N_C) \arrow[r, "pr"] & H^0(L^*_{|C}) \arrow[r] & 0 \end{tikzcd} , \tag{3-4}$$

we have $\underline{\imath}(u_{k-1}) = x \otimes (y \cdot u_{k-1}) - y \otimes (x \cdot u_{k-1})$, and $\underline{pr}(u_1 \otimes v_k) = u_1 \cdot v_k$ for all $u_i, v_i \in \mathcal{V}_i$. Here, we used the natural identifications $H^0(\mathcal{O}(n)) \cong \mathcal{V}_n$ and $H^0(\mathcal{O}(n) \oplus \mathcal{O}(n)) \cong \mathcal{V}_{1,n}$ for any integer $n \geq 0$.

Let us define the vector subspaces $\mathcal{V}'$ and $\mathcal{V}''$ of $\mathcal{V}_{1,k}$ by

$$\mathcal{V}' := \{x \otimes u_x + y \otimes u_y \mid u \in \mathcal{V}_{k+1}\},$$

where u_x and u_y denote partial derivatives, and

$$\mathcal{V}'' := \underline{\imath}(\mathcal{V}_{k-1}) = \{x \otimes (y \cdot u) - y \otimes (x \cdot u) \mid u \in \mathcal{V}_{k-1}\}.$$

Then $\mathcal{V}_{1,k} = \mathcal{V}' \oplus \mathcal{V}''$ is easily verified.

Proposition 3.4. *Let $C \in \mathcal{Y} \subseteq \mathcal{Z}$ be a rational contact curve and consider bundle isomorphisms ϕ_C, ϕ'_C and ϕ''_C which induce the commutative diagrams* (3-3) *and* (3-4)*. Then we have*

$$\phi_C^*(\mathcal{V}') = T_C\mathcal{Y} \quad \textit{and} \quad \phi_C^*(\mathcal{V}'') = \imath(H^0(L^{\perp}_{|C}/\tau)). \tag{3-5}$$

Proof. The map $\eta : \mathcal{V}_{k+1} \to \mathcal{V}_{1,k}$ given by $\eta(u) := \frac{1}{k+1}(x \otimes u_x + y \otimes u_y)$ splits the top exact sequence of (3-4). Thus, Corollary 3.3. implies that there is a map $\delta : \mathcal{V}_{k+1} \to \mathcal{V}_{k-1}$ such that $T_C\mathcal{Y} = \phi_C^*\left(\{(\eta + \underline{\imath} \circ \delta)(u) \mid u \in \mathcal{V}_{k+1}\}\right)$.

Since $\mathcal{V}_{k+1}^* \otimes \mathcal{V}_{k-1} \cong \mathcal{V}_{2k} \oplus \mathcal{V}_{2k-2} \oplus \ldots \oplus \mathcal{V}_2$ it follows that there are polynomials $v_i \in \mathcal{V}_i$, $i = 2, 4, \ldots 2k$ such that

$$\delta(u) = \langle u, v_{2k} \rangle_{k+1} + \langle u, v_{2k-2} \rangle_k + \ldots + \langle u, v_2 \rangle_2 .$$

From Lemma 3.2. and an easy calculation we conclude that δ must satisfy the following condition:

$$\text{if } r^2 | u \text{ for some } r \in \mathcal{V}_1,\ u \in \mathcal{V}_{k+1} \text{ then } r | \delta(u). \tag{3-6}$$

Using the $Sl(2, \mathbb{C})$-equivariance of $\langle \_, \_ \rangle$ we compute that for any $r \in \mathcal{V}_1$, $r|\delta(r^{k+1})$ if and only if $r|v_{2k}$. Thus, (3-6) implies that every $r \in \mathcal{V}_1$ divides v_{2k}, hence $v_{2k} = 0$.

Next, a similar calculation shows that $r|\delta(r^k s)$ for all $s \in \mathcal{V}_1$ if and only if $r|v_{2k-2}$. Again, this together with (3-6) implies $v_{2k-2} = 0$.

Continuing with similar arguments, we see successively that $v_{2k} = v_{2k-2} = \ldots = v_2 = 0$, thus $\delta = 0$, and this shows the first equation in (3-5). The second equation is immediate from the commutativity of (3-4). $\square$

Proposition 3.5. *Let $F_{\mathcal{Y}} := \pi^{-1}(\mathcal{Y})$ with the principal G-bundle $\pi : F \to \mathcal{Z}$ from* (2-1) *and let $\pi_{\mathcal{Y}} : \mathfrak{F}_{\mathcal{Y}} \to \mathcal{Y}$ denote the total $\mathcal{V}_{k+1}$-coframe bundle of $\mathcal{Y}$. The set*

$$\hat{F} := \left\{ \phi_C : \mathcal{O}(k) \oplus \mathcal{O}(k) \to N_C \;\middle|\; \begin{array}{c} C \in \mathcal{Y},\ \phi_C \textit{ a bundle isomorphism} \\ \textit{which satisfies } (3\text{-}5). \end{array} \right\} \subseteq F_{\mathcal{Y}}$$

is a reduction of $F_{\mathcal{Y}}$ with structure group

$$G^{\Delta} := \{(c, A, A) \mid c \in \mathbb{C}^*, A \in Sl(2, \mathbb{C})\} \subseteq G.$$

Moreover, the map

$$\begin{aligned} \zeta : \ \hat{F} &\to \mathfrak{F}_{\mathcal{Y}} \\ \phi_C &\mapsto (\underline{pr} \circ (\phi_C^*)^{-1} : T_C\mathcal{Y} \to \mathcal{V}_{k+1}) \end{aligned}$$

is an imbedding, and the image $\zeta(\hat{F}) \subseteq \mathfrak{F}_{\mathcal{Y}}$ is a G_{k+1}-structure on $\mathcal{Y}$.

Proof. The proof is straightforward: first of all, by our previous discussion we know that $\pi^{-1}(C) \cap \hat{F} \neq \emptyset$ for all $C \in \mathcal{Y}$. Moreover, if $\phi_C^1, \phi_C^2 \in \hat{F}_C$, then $\psi := (\phi_C^1)^{-1} \circ \phi_C^2 \in Aut(\mathcal{O}(k) \oplus \mathcal{O}(k))$ must satisfy $\psi^*(\mathcal{V}') = \mathcal{V}'$ and $\psi^*(\mathcal{V}'') = \mathcal{V}''$. This is the case precisely if $\psi \in G^\Delta$. Thus $\hat{F}$ is a G^Δ-reduction of $F_{\mathcal{Y}}$.

The verification of the stated properties of ζ is left to the reader. □

By abuse of notation, we shall identify $\hat{F}$ with $\zeta(\hat{F})$ and thus regard $\hat{F}$ as a G_{k+1}-structure on $\mathcal{Y}$. The tautological 1-form of $\pi : \hat{F} \to \mathcal{Y}$ is then given by

$$\hat{\theta} = \underline{pr} \circ (\theta|_{\hat{F}}).$$

The decomposition $\mathcal{V}_{1,k} = \mathcal{V}' \oplus \mathcal{V}''$ induces the decomposition

$$\mathcal{V}_{1,k}^* \otimes \mathcal{V}_{1,k} = (\mathcal{V}'^* \otimes \mathcal{V}') \oplus (\mathcal{V}'^* \otimes \mathcal{V}'') \oplus (\mathcal{V}''^* \otimes \mathcal{V}') \oplus (\mathcal{V}''^* \otimes \mathcal{V}'').$$

Projection onto the first direct summand composed with conjugation by $\underline{pr}|_{\mathcal{V}'}$ yields a homomorphism

$$p : \mathfrak{gl}(\mathcal{V}_{1,k}) \to \mathfrak{gl}(\mathcal{V}_{k+1}).$$

It is not hard to verify that the 1-form

$$\hat{\omega} := (p \circ \omega)|_{\hat{F}}$$

yields a connection on $\pi : \hat{F} \to \mathcal{Y}$.

Definition 3.6.

(1) A $G_{1,k}$-connection is a triple $(\pi : F \to \mathcal{Z}, \theta, \omega)$ of a $G_{1,k}$-reduction F of $\mathcal{Z}$ and the tautological and connection 1-forms θ and ω.
(2) Likewise, a G_{k+1}-connection is a triple $(\pi : \hat{F} \to \mathcal{Y}, \hat{\theta}, \hat{\omega})$ of a G_{k+1}-reduction $\hat{F}$ of $\mathcal{Y}$ and the tautological and connection 1-forms $\hat{\theta}$ and $\hat{\omega}$.
(3) Suppose $(\pi : F \to \mathcal{Z}, \theta, \omega)$ and $(\pi : \hat{F} \to \mathcal{Y}, \hat{\theta}, \hat{\omega})$ are a $G_{1,k}$-connection and a G_{k+1}-connection respectively and suppose there is an injective bundle map

$$\begin{array}{ccc} \hat{F} & \xrightarrow{\jmath} & F \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \mathcal{Y} & \xrightarrow{\jmath} & \mathcal{Z} \end{array}$$

such that $\hat{\theta} = \underline{pr} \circ \theta|_{\hat{F}}$ and $\hat{\omega} = p \circ \omega|_{\hat{F}}$.

Then $(\pi : \hat{F} \to \mathcal{Y}, \hat{\theta}, \hat{\omega})$ is called a *restriction* of $(\pi : F \to \mathcal{Z}, \theta, \omega)$, whereas $(\pi : F \to \mathcal{Z}, \theta, \omega)$ is called an *extension* of $(\pi : \hat{F} \to \mathcal{Y}, \hat{\theta}, \hat{\omega})$.

The following Proposition is straightforward and the proof is omitted.

Proposition 3.7. *If the $G_{1,k}$-connection $(\pi : F \to \mathcal{Z}, \theta, \omega)$ is an extension of the G_{k+1}-connection $(\hat{\pi} : \hat{F} \to \mathcal{Y}, \hat{\theta}, \hat{\omega})$ and if Θ and $\hat{\Theta}$ denote the torsion of ω and $\hat{\omega}$ respectively then*

$$\hat{\Theta} = \underline{pr} \circ \Theta|_{\hat{F}}. \quad \square$$

Definition 3.8. A connection on $\pi_{\mathcal{Y}} : \hat{F} \to \mathcal{Y}$ is called *special* if it is the restriction of a special connection on $\pi : F \to \mathcal{Z}$.

Of course, from our discussion preceding Definition 3.6. we know that if $\mathcal{Z}$ is the moduli space of rational curves in $\mathcal{W}$ whose normal bundle is equivalent to $\mathcal{O}(k) \oplus \mathcal{O}(k)$ and if $\mathcal{Y} \subseteq \mathcal{Z}$ is the subset of contact curves then every connection on $\pi : F \to \mathcal{Z}$ has a restriction to the G_{k+1}-structure $\pi : \hat{F} \to \mathcal{Y}$.

Let us now investigate the *intrinsic torsion* of both the $G_{1,k}$-structure $\pi : F \to \mathcal{Z}$ and the G_{k+1}-structure $\pi_{\mathcal{Y}} : \hat{F} \to \mathcal{Y}$. The Spencer sequence (1-1) reads

(3-7)
$$0 \longrightarrow \mathfrak{g}_{1,k}^{(1)} \longrightarrow \mathcal{V}_{1,k}^* \otimes \mathfrak{g}_{1,k} \xrightarrow{Sp} \Lambda^2 \mathcal{V}_{1,k}^* \otimes \mathcal{V}_{1,k} \longrightarrow H^{0,2}(\mathfrak{g}_{1,k}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{g}_{k+1}^{(1)} \longrightarrow \mathcal{V}_{k+1}^* \otimes \mathfrak{g}_{k+1} \xrightarrow{Sp} \Lambda^2 \mathcal{V}_{k+1}^* \otimes \mathcal{V}_{k+1} \longrightarrow H^{0,2}(\mathfrak{g}_{k+1}) \longrightarrow 0.$$

Lemma 3.9. *If $k \geq 2$ then $\mathfrak{g}_{1,k}^{(1)} = 0$ and $\mathfrak{g}_{k+1}^{(1)} = 0$.*

Proof. Let $\varphi \in \mathfrak{g}_{1,k}^{(1)}$. We regard φ as a linear map $\varphi : \mathcal{V}_{1,k} \to \mathfrak{g}_{1,k}$. Pick two arbitrary bases (r_1, r_2) and (s_1, s_2) of $\mathcal{V}_1$. Then the set $\{r_i \otimes s_1^{k-j} s_2^j \mid i = 1, 2, j = 0, \ldots k\}$ forms a basis of $\mathcal{V}_{1,k}$. We have

$$\left(\varphi(r_1 \otimes s_1^k)\right)(r_2 \otimes s_2^k) - \left(\varphi(r_2 \otimes s_2^k)\right)(r_1 \otimes s_1^k) = 0.$$

If we let $\varphi(r_i \otimes s_i^k) := (A_i, B_i)$ for $i = 1, 2$ be the decomposition in $\mathfrak{g}_{1,k} \cong \mathfrak{gl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, then this equation reads

$$((A_1 \cdot r_2) \otimes s_2^k) + (r_2 \otimes (B_1 \cdot s_2^k)) - ((A_2 \cdot r_1) \otimes s_1^k) - (r_1 \otimes (B_2 \cdot s_1^k)) = 0. \tag{3-8}$$

Note that $B_1 \cdot s_2^k \in span\{s_1 s_2^{k-1}, s_2^k\}$. Taking the $(r_2 \otimes s_1 s_2^{k-1})$-component of (3-8) w.r.t. the above basis we conlude that s_2^k is an eigenvector of B_1. Since this is true for *any* s_2 which is linearly independent of s_1, it follows that B_1 is a multiple of the identity. On the other hand, $trace(B_1) = 0$. Thus, we have $B_1 = 0$.

Likewise, $A_1 = 0$, hence $\varphi(r_1 \otimes s_1^k) = 0$ for arbitrary $r_1, s_1 \in \mathcal{V}_1$. Since elements of this form span all of $\mathcal{V}_{1,k}$, $\varphi = 0$ follows.

The proof of the second statement is of similar nature but simpler. We omit the details. $\square$

To calculate the irreducible components of (3-7), note that as a G-module, $\mathfrak{g}_{1,k} \cong \mathcal{V}_{0,0} \oplus \mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}$. In fact, the equivalence is determined by the equation

$$(p_{0,0} + p_{2,0} + p_{0,2}) \cdot q_{1,k} := \langle p_{0,0}, q_{1,k} \rangle_{0,0} + \langle p_{2,0}, q_{1,k} \rangle_{1,0} + \langle p_{0,2}, q_{1,k} \rangle_{0,1} \tag{3-9}$$

for all $p \in \mathcal{V}$ and $q \in \mathcal{V}$

For the rest of this section we shall assume that $k = 2$. In this case, a calculation shows that the decomposition of the Spencer sequence (3-7) into irreducible submodules is

$$0 \longrightarrow \begin{matrix} \mathcal{V}_{1,0} \oplus 3\mathcal{V}_{1,2} \oplus \mathcal{V}_{1,4} \\ \oplus \mathcal{V}_{3,2} \end{matrix} \xrightarrow{Sp} \begin{matrix} \mathcal{V}_{1,0} \oplus 3\mathcal{V}_{1,2} \oplus 2\mathcal{V}_{1,4} \oplus \mathcal{V}_{1,6} \\ \oplus \mathcal{V}_{3,0} \oplus \mathcal{V}_{3,2} \oplus \mathcal{V}_{3,4} \end{matrix} \longrightarrow \begin{matrix} \mathcal{V}_{1,4} \oplus \mathcal{V}_{1,6} \\ \oplus \mathcal{V}_{3,0} \oplus \mathcal{V}_{3,4} \end{matrix} \longrightarrow 0.$$

More explicitly, if $\varphi \in \mathcal{V}_{1,2}^* \otimes \mathfrak{g}_{1,2}$, then there are elements $r_{i,j}, r'_{i,j}, r''_{i,j} \in \mathcal{V}_{i,j}$ such that for $p_{1,2} \in \mathcal{V}_{1,2}$,

(3-10)
$$\begin{aligned} \varphi(p_{1,2}) = &\Big(\langle r_{1,2}, p_{1,2} \rangle_{1,2} \Big) + \Big(\langle r_{3,2}, p_{1,2} \rangle_{1,2} + \langle r'_{1,2}, p_{1,2} \rangle_{0,2} \Big) \\ &+ \Big(\langle r_{1,4}, p_{1,2} \rangle_{1,2} + \langle r''_{1,2}, p_{1,2} \rangle_{1,1} + \langle r_{1,0}, p_{1,2} \rangle_{1,0} \Big) \in \mathcal{V}_{0,0} \oplus \mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}. \end{aligned}$$

Likewise, for any $T \in \Lambda^2 \mathcal{V}_{1,2}^* \otimes \mathcal{V}_{1,2}$, there are elements $s_{i,j}, s'_{i,j}, s''_{i,j} \in \mathcal{V}_{i,j}$ such that for all $p, q \in \mathcal{V}_{1,2}$,

(3-11)
$$\begin{aligned} T(p,q) = &\Big\langle s_{1,2}, \langle p,q \rangle_{1,0} \Big\rangle_{0,2} + \Big\langle s_{1,4}, \langle p,q \rangle_{1,0} \Big\rangle_{0,3} + \Big\langle s_{1,6}, \langle p,q \rangle_{1,0} \Big\rangle_{0,4} \\ &+ \Big\langle s_{1,0}, \langle p,q \rangle_{0,1} \Big\rangle_{1,0} + \Big\langle s'_{1,2}, \langle p,q \rangle_{0,1} \Big\rangle_{1,1} + \Big\langle s'_{1,4}, \langle p,q \rangle_{0,1} \Big\rangle_{1,2} \\ &+ \Big\langle s_{3,0}, \langle p,q \rangle_{0,1} \Big\rangle_{2,0} + \Big\langle s_{3,2}, \langle p,q \rangle_{0,1} \Big\rangle_{2,1} + \Big\langle s_{3,4}, \langle p,q \rangle_{0,1} \Big\rangle_{2,2} \\ &+ \Big\langle s''_{1,2}, \langle p,q \rangle_{1,2} \Big\rangle_{0,0}. \end{aligned}$$

Using the tuple $(s_{1,2}, s_{1,4}, s_{1,6}, s_{1,0}, s'_{1,2}, s'_{1,4}, s_{3,0}, s_{3,2}, s_{3,4}, s''_{1,2})$ as coordinates for $\Lambda^2 \mathcal{V}_{1,2}^* \otimes \mathcal{V}_{1,2}$, another calculation shows that
(3-12)
$$\begin{aligned} Sp(\varphi) = \Big(&-\frac{1}{6}(r_{1,2} - 3r'_{1,2} + 4r''_{1,2}),\ -\frac{1}{2} r_{1,4},\ 0,\ r_{1,0},\ -\frac{1}{8}(r_{1,2} + r'_{1,2} - 4r''_{1,2}), \\ &-\frac{1}{2} r_{1,4},\ 0,\ -\frac{1}{4} r_{3,2},\ 0,\ -\frac{1}{3}(r_{1,2} - 3r'_{1,2} - 8r''_{1,2}) \Big), \end{aligned}$$

where the $r_{i,j}$'s are determined by φ as in (3-10).

Lemma 3.10. *Let $\mathcal{Z}$ be as before, and suppose that ω_1 is a special connection on $\mathcal{Z}$. Then another connection ω_2 on $\mathcal{Z}$ is special if and only if there are functions $r_{i,j}, r'_{i,j}$ on F with values in $\mathcal{V}_{i,j}$ such that*

$$\omega_2 = \omega_1 + \Big(\langle r_{1,2}, \theta \rangle_{1,2} \Big) + \Big(\langle r_{3,2}, \theta \rangle_{1,2} + \langle r'_{1,2}, \theta \rangle_{0,2} \Big) + \Big(\langle r_{1,0}, \theta \rangle_{1,0} \Big). \tag{3-13}$$

Here, we use the identification (3-9) *to regard the ω_i's as $\mathcal{V}_{0,0} \oplus \mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}$-valued 1-forms on F.*

Proof. First of all, we define the vector bundles $\mathcal{V}^F_{i,j} := F \times_G \mathcal{V}_{i,j}$ over $\mathcal{Z}$. Recall from §1 that the difference between two connections on $F \to \mathcal{Z}$ is determined by a

section of the bundle $T^*\mathcal{Z}\otimes\mathfrak{g}_F$. Since by (3-9) we have $\mathfrak{g}_F\cong\mathcal{V}^F_{0,0}\oplus\mathcal{V}^F_{2,0}\oplus\mathcal{V}^F_{0,2}$, we can decompose

$$T^*\mathcal{Z}\otimes\mathfrak{g}_F=B_1^F\oplus B_2^F \quad \text{with} \quad \begin{cases} B_1^F := T^*\mathcal{Z}\otimes\left(\mathcal{V}^F_{0,0}\oplus\mathcal{V}^F_{2,0}\right), & \text{and} \\ B_2^F := T^*\mathcal{Z}\otimes\mathcal{V}^F_{0,2}. \end{cases}$$

Note that $B_2^F\cong\mathcal{V}^F_{1,2}\otimes\mathcal{V}^F_{0,2}\cong\mathcal{V}^F_{1,0}\oplus\mathcal{V}^F_{1,2}\oplus\mathcal{V}^F_{1,4}$.

Consider the vector bundle

$$\Delta := \bigcup_{t\in\mathcal{Z}} H^0(Hom(N_{C_t},\tau_{C_t}))\to\mathcal{Z}.$$

For a fixed $t\in\mathcal{Z}$, $Hom(N_{C_t},\tau_{C_t})\cong Hom(\mathcal{O}(2)\oplus\mathcal{O}(2),\mathcal{O}(2))\cong\mathcal{O}(0)\oplus\mathcal{O}(0)$, hence $H^0(Hom(N_{C_t},\tau_{C_t}))\cong\mathcal{V}_{1,0}$ as a G-module, and thus $\Delta\cong\mathcal{V}^F_{1,0}$.

Let us fix a special connection $\omega_0=\tilde\omega+\hat\sigma_0$ with some section σ_0 of the split bundle $S\to\mathcal{Z}$. Given a local section δ of Δ, we let $\omega:=\tilde\omega+\hat\sigma$ with $\sigma:=\sigma_0+\delta$, and define $\psi(\delta):=\omega-\omega_0$. From Lemma 2.4. it is easy to verify that $\psi(\delta)$ is a local section of $B_2^F\subseteq T^*\mathcal{Z}\otimes\mathfrak{g}_F$, that the correspondence $\delta\mapsto\psi(\delta)$ determines a bundle map $\psi:\Delta\to B_2^F$, and that ψ is independent of the choice of ω_0. Also, it is obvious that ψ is non-vanishing, hence by Schur's Lemma $\psi(\Delta)=\mathcal{V}^F_{1,0}\subseteq B_2^F$.

Let $\omega_1=\tilde\omega_1+\hat\sigma_1$ be the decomposition of the special connection ω_1 where σ_1 is a section of the split bundle S. Then ω_2 is *special* if and only if $\omega_2=\tilde\omega_2+\hat\sigma_2$ for some section σ_2 of S, if and only if $\omega_2-\omega_1=(\tilde\omega_2-\tilde\omega_1)+\psi(\delta)$ where $\delta:=\sigma_2-\sigma_1$ is a section of Δ, if and only if $\omega_2-\omega_1$ is a section of $B_1^F\oplus\psi(\Delta)=B_1^F\oplus\mathcal{V}^F_{1,0}\subseteq T^*\mathcal{Z}\otimes\mathfrak{g}_F$.

Comparing (3-13) with (3-10), we see that this is satisfied if and only if (3-13) holds. □

Theorem 3.11. *Let $\mathcal{Z}$ be the moduli space of rational curves in a 3-fold $\mathcal{W}$ whose normal bundle is equivalent to $\mathcal{O}(2)\oplus\mathcal{O}(2)$, and let $\pi:F\to\mathcal{Z}$ be the associated $G_{1,2}$-structure. Then there is a unique connection ω on F and a function $s_{3,0}:F\to\mathcal{V}_{3,0}$ such that the torsion of ω is given by*

$$\Theta=\Big\langle s_{3,0}(u),\langle\theta,\theta\rangle_{0,1}\Big\rangle_{2,0}.$$

Moreover, ω is a special connection.

This simple form of the torsion is quite remarkable; indeed, Lemma 3.9. implies that for $k=2$, $rank(H_F^{0,2})=48$. Thus, Theorem 3.11. says that *most* of the intrinsic torsion of F vanishes.

Proof. First of all, note that it suffices to prove the Theorem locally, i.e. we need to show that F can be covered by open sets $\mathcal{U}_i$ on which a connection ω_i with the stated properties exists. Then, by uniqueness, ω_i and ω_j must coincide on $\mathcal{U}_i\cap\mathcal{U}_j$, thus ω_i is the restriction to $\mathcal{U}_i$ of a connection ω defined on *all* of F. In the proof, we will replace $\mathcal{U}_i$ by $\mathcal{Z}$ and thus we may assume that all local properties of $\mathcal{Z}$ hold globally.

By Proposition 2.6. we can find a *special* connection ω_0 on F. Then there are functions $(s_{1,2},s_{1,4},\dots,s''_{1,2})$ on F with values in $\mathcal{V}_{1,2},\mathcal{V}_{1,4},\dots,\mathcal{V}_{1,2}$ respectively such that the torsion Θ_0 of ω_0 is given by $\Theta_0(\xi,\xi')=T_0(p,q)$ with $p=\theta(\xi),q=\theta(\xi')$, and where $T_0(p,q)$ is determined by the $s_{\dots}$'s as in (3-11).

We call $T_0 : F \to \Lambda^2\mathcal{V}_{1,2}^* \otimes \mathcal{V}_{1,2}$ the *torsion map of* ω_0. By abuse of notation, we write $T_0 = s_{1,2} + s_{1,4} + \ldots + s''_{1,2}$.

We shall say that $r \in \mathcal{V}_1$ *divides* $p \in \mathcal{V}_{1,k}$ and write $r|p$ if $p = x \otimes p_1 + y \otimes p_2$ and r divides both p_1 and p_2. From Corollary 2.8. we obtain the following criterion:

(3-14) If for $r \in \mathcal{V}_1$ and $p, q \in \mathcal{V}_{1,2}$ we have $r|p$ and $r|q$ then also $r|T_0(u)(p, q)$.

From (3-14), we can conclude that some of the $s_{i,j}$'s must vanish identically. For example, let $p := x \otimes r^2$ and $q := y \otimes r^2$ for $r \in \mathcal{V}_1$. Then $\langle p, q\rangle_{0,1} = \langle p, q\rangle_{1,2} = 0$ and $\langle p, q\rangle_{1,0} = 1 \otimes r^4 \in \mathcal{V}_{0,4}$. Since $r|p$ and $r|q$, (3-14) implies that $r|T_0(u)(p, q)$. An easy computation shows that this is the case if and only if $r|s_{1,6}(u)$. But this must be true for *all* $r \in \mathcal{V}_1$ and $u \in F$, thus we conclude $s_{1,6} = 0$.

By similar calculations we see that (3-14) is satisfied if and only if

$$s_{1,4} = s'_{1,4} = s_{1,6} = s_{3,4} = s''_{1,2} - 2s_{1,2} = 0. \tag{3-15}$$

From here it follows that the intrinsic torsion of the $G_{1,2}$-structure is represented by $s_{3,0}$. Also, from (3-12) and (3-15) it follows that $T_0 - s_{3,0} = Sp(\varphi)$ for some function $\varphi : F \to \mathcal{V}_{1,2}^* \otimes \mathfrak{g}_{1,2}$ whose $r_{1,4}$ and $r''_{1,2}$ component vanish. Then Lemma 3.10. implies that the connection $\omega := \omega_0 - \varphi$ is still *special*, and if we let T be the torsion map of ω then by (1-7) we have $T = T_0 - Sp(\varphi) = s_{0,3}$. Thus, the torsion of the special connection ω is of the desired form.

The uniqueness follows from Proposition 1.2. together with Lemma 3.9. □

Definition 3.12. The unique special connection from Theorem 3.11. is called the *intrinsic connection of* $\mathcal{Z}$.

Theorem 3.13. *Let $\mathcal{W}$ be a contact 3-fold with contact line bundle $L \to \mathcal{W}$, and let $\mathcal{Y}$ be the moduli space of rational contact curves C such that $L|_C \cong \mathcal{O}(-3)$. Then there is a unique torsion free connection $\hat{\omega}$ on the G_3-structure $\pi_{\mathcal{Y}} : \hat{F} \to \mathcal{Y}$. Moreover, $\hat{\omega}$ is special.*

Proof. By Proposition 3.1. $\mathcal{Y} \subseteq \mathcal{Z}$ is a submanifold with $\mathcal{Z}$ as in Theorem 3.11. Let ω_0 be the intrinsic connection on $\mathcal{Z}$, and let $s_3 : F \to \mathcal{V}_3$ be such that $s_3 \otimes 1 : F \to \mathcal{V}_{3,0}$ is the torsion function of ω_0 from Theorem 3.11.

Let $s_{1,2} := x \otimes (s_3)_x + y \otimes (s_3)_y \in \mathcal{V}_{1,2}$, and let $\varphi : F \to \mathcal{V}_{1,2}^* \otimes \mathfrak{g}_{1,2}$ be determined by (3-10) with $r_{1,2} = r'_{1,2} := 2s_{1,2}$, all other $r_{i,j}$'s $= 0$. As before, φ can be regarded as a section of $T^*\mathcal{Z} \otimes \mathfrak{g}_F$, and by Lemma 3.10, the connection $\omega := \omega_0 + \varphi$ is again a special connection on $\mathcal{Z}$. We denote by $\hat{\omega}$ the restriction of ω to $\mathcal{Y}$, and let Θ, Θ_0 and $\hat{\Theta}$ denote the torsion forms of ω, ω_0 and $\hat{\omega}$ respectively. By (1-7), $\Theta = \Theta_0 + Sp(\varphi)$.

Then by Proposition 3.7. and some calculation we have

$$\begin{aligned}
\hat{\Theta} &= \underline{pr} \circ \Theta \\
&= \underline{pr}\left(\begin{array}{c} \left\langle s_{3,0}, \langle\theta,\theta\rangle_{0,1}\right\rangle_{2,0} - \frac{1}{2}\left\langle s_{1,2}, \langle\theta,\theta\rangle_{0,1}\right\rangle_{1,1} \\ +\frac{2}{3}\left\langle s_{1,2}, \langle\theta,\theta\rangle_{1,0}\right\rangle_{0,2} + \frac{4}{3}\left\langle s_{1,2}, \langle\theta,\theta\rangle_{1,2}\right\rangle_{0,0} \end{array} \right) \\
&= 0.
\end{aligned}$$

Thus, $\hat{\omega}$ is the desired torsion free special connection. The uniqueness follows from Proposition 1.2. together with Lemma 3.9. □

§4 Torsion free $G_{1,2}$-structures.

In this entire section, we shall consider complex sixfolds $\mathcal{Z}$ which carry a *torsion free* $G_{1,2}$-structure $\pi : F \to \mathcal{Z}$. In this case, there is a unique torsion free connection $\omega = \omega_{0,0} + \omega_{2,0} + \omega_{0,2}$ on F where $\omega_{i,j}$ takes values in $\mathcal{V}_{i,j}$. Here, we used the identification $\mathfrak{g}_{1,2} \cong \mathcal{V}_{0,0} \oplus \mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}$ from (3-9).

For convenience, we shall define the pairings

$$\begin{array}{rccl} \langle\!\langle -, - \rangle\!\rangle^{(k)} : & (\mathcal{V}_{0,0} \oplus \mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}) \otimes \mathcal{V}_{i,j} & \to & \mathcal{V}_{i,j} \\ & \langle\!\langle p_{0,0} + p_{2,0} + p_{0,2},\ q \rangle\!\rangle^{(k)} & := & k \langle p_{0,0}, q \rangle_{0,0} + \langle p_{2,0}, q \rangle_{1,0} + \langle p_{0,2}, q \rangle_{0,1} \end{array}$$

Then the *first structure equation* of ω reads

$$d\theta + \langle\!\langle \omega, \theta \rangle\!\rangle^{(1)} = 0 \tag{4-1}$$

with the $\mathcal{V}_{1,2}$-valued tautological 1-form θ.

Moreover, the *curvature 2-form* Ω takes values in $\mathfrak{g}_{1,2} \cong \mathcal{V}_{0,0} \oplus \mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}$, and is defined as

$$\begin{aligned} \Omega &= d\omega + \omega \wedge \omega \\ &= d\omega - \frac{1}{2} \left(\langle \omega_{2,0}, \omega_{2,0} \rangle_{1,0} + \langle \omega_{0,2}, \omega_{0,2} \rangle_{0,1} \right). \end{aligned} \tag{4-2}$$

Differentiating (4-1), we obtain the *first Bianchi identity*

$$\langle\!\langle \Omega, \theta \rangle\!\rangle^{(1)} = 0. \tag{4-3}$$

Let $\mathbf{K}(\mathfrak{g}_{1,2})$ be given by the exact sequence

$$0 \longrightarrow \mathbf{K}(\mathfrak{g}_{1,2}) \longrightarrow \Lambda^2 \mathcal{V}_{1,2}^* \otimes \mathfrak{g}_{1,2} \xrightarrow{Sp_2} \Lambda^3 \mathcal{V}_{1,2}^* \otimes \mathcal{V}_{1,2},$$

where Sp_2 is given by skew-symmetrization of $\Lambda^2 \mathcal{V}_{1,2} \otimes \mathfrak{g}_{1,2} \subseteq \Lambda^2 \mathcal{V}_{1,2} \otimes (\mathcal{V}_{1,2}^* \otimes \mathcal{V}_{1,2})$. The first Bianchi identity (4-3) can be interpreted as stating that Ω is a section of $F \times_G \mathbf{K}(\mathfrak{g}_{1,2})$.

A calculation shows that, as a G-module, $\mathbf{K}(\mathfrak{g}_{1,2}) \cong \mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}$. More explicitly, there is a function $\mathbf{a} = a_{2,0} + a_{0,2} : F \to \mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}$ such that

$$\begin{aligned} \Omega = &\left(-4 \left\langle a_{2,0}, \langle \theta, \theta \rangle_{1,2} \right\rangle_{0,0} + 3 \left\langle a_{0,2}, \langle \theta, \theta \rangle_{0,1} \right\rangle_{0,2} \right) \\ &+ \left(\left\langle a_{2,0}, \langle \theta, \theta \rangle_{0,1} \right\rangle_{2,0} + \left\langle a_{0,2}, \langle \theta, \theta \rangle_{1,0} \right\rangle_{0,2} - 7 \left\langle a_{0,2}, \langle \theta, \theta \rangle_{1,2} \right\rangle_{0,0} \right). \end{aligned} \tag{4-4}$$

This implies, in particular, that $d\omega_{0,0} = 0$. Therefore, by the *Ambrose-Singer-Holonomy Theorem* [KN], the holonomy of ω is contained in the subgroup

$$H_{1,2} := G_{1,2} \cap Sl(V_{1,2}).$$

Taking the derivative of (4-4) and solving for $d\mathbf{a}$, we see that there is a function $\mathbf{b} : F \to \mathcal{V}_{1,2}$ such that

$$d\mathbf{a} = \langle\!\langle \omega, \mathbf{a} \rangle\!\rangle^{(-2)} + 3 \langle \mathbf{b}, \theta \rangle + \langle \mathbf{b}, \theta \rangle \tag{4-5}$$

Once again, we take the derivative of (4-5) and solve for $d\mathbf{b}$. We see that there is a function $c : F \to V_{0,0}$ such that

$$\begin{aligned} d\mathbf{b} = \langle\!\langle \mathbf{b}, \omega \rangle\!\rangle^{(-3)} + 2\left\langle \langle a_{2,0}, a_{0,2}\rangle_{0,0}, \theta \right\rangle_{1,1} + \left\langle \langle a_{0,2}, a_{0,2}\rangle_{0,0}, \theta \right\rangle_{0,2} \\ + \left\langle -\frac{4}{3}\langle a_{2,0}, a_{2,0}\rangle_{2,0} - 7\langle a_{0,2}, a_{0,2}\rangle_{0,2} + c, \theta \right\rangle_{0,0} \end{aligned} \tag{4-6}$$

Taking exterior derivatives one more time and solving for dc we calculate that

$$dc = -4c\omega_{0,0}. \tag{4-7}$$

The reader who is familiar with [Br] will note the similarity of the structure equations (4-1) - (4-7) with the structure equations for H_3-connections where $H_3 = G_3 \cap Sl(\mathcal{V}_3)$. This is by no means a coincidence. As we shall see in the following section, there is a close relationship between $H_{1,2}$-structures and H_3-structures.

Let $F_0 \subseteq F$ be an integral hypersurface of $\omega_{0,0}$, i.e. a hypersurface such that $\omega_{0,0}|_{F_0} \equiv 0$. Then F_0 is a torsion free $H_{1,2}$-reduction of F. We shall denote the restrictions of θ, $\omega_{2,0}$, $\omega_{0,2}$, $\mathbf{a}$ and $\mathbf{b}$ to F_0 by the same letters. Note that $\omega_0 := \omega|_{F_0} = \omega_{2,0} + \omega_{0,2}$ is $\mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}$-valued. Also, by (4-7), c is *constant* on F_0.

Definition 4.1. Let $\pi : F \to \mathcal{Z}$ be a torsion free $G_{1,2}$-structure and let $F_0 \subseteq F$ be an integral hypersurface of $\omega_{0,0}$. Then F_0 is called an *associated $H_{1,2}$-structure of F*.

The choice of associated $H_{1,2}$-structures is, of course, not unique. However, given two such structures F_0 and F_0' then $F_0' = R_{tI} \cdot F_0$ for some $t \in \mathbb{C}^*$. *Hence, to each torsion free $G_{1,2}$-structure there is a one parameter family of associated $H_{1,2}$-structures.*

Our approach to solve the structure equations (4-1) - (4-7) will be motivated by the steps pursued in [Br] to solve the structure equations of an H_3-connection.

Let

$$K := \mathbf{a} + \mathbf{b} : F_0 \to \mathcal{V}, \quad \text{where} \quad \mathcal{V} = \mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2} \oplus \mathcal{V}_{1,2}.$$

Equations (4-5) - (4-6) can be summarized as

$$dK = J(\theta + \omega_0)$$

where J is a function on F with values in $Hom(\mathcal{V}, \mathcal{V})$. Now $J = K^*(\mathbf{J}_c)$ where $\mathbf{J}_c : \mathcal{V} \to Hom(\mathcal{V}, \mathcal{V})$ is a polynomial mapping which depends upon a parameter c. If we write $\mathbf{J}_c$ relative to the standard basis of $\mathcal{V}$ then it has a 12×12-matrix representation whose entries are polynomials in the components of $\mathbf{a}$ and $\mathbf{b}$.

It turns out that this matrix $\mathbf{J}_c$ is not invertible. In fact, generically the rank of $\mathbf{J}_c$ is calculated to be 10. This implies that the image of K is contained in some 10-dimensional subvariety of $\mathcal{V}$.

Let us this once comment on the mechanical calculations which are performed to arrive at this conclusion. The attempt of simply taking the determinant of $\mathbf{J}_c$ on `MATHEMATICA` failed miserably at first: after more than 10 minutes of calculation, memory overflows occured.

The next approach was to use the $H_{1,2}$-equivariance of $\mathbf{J}_c$. Under the generic assumption that both $a_{2,0}$ and $a_{0,2}$ are not squares of a linear polynomial, we may

assume that $a_{2,0} = txy \otimes 1$ and $a_{0,2} = t'xy \otimes 1$ for some $t, t' \in \mathbb{C}$. Making this replacement simplifies $\mathbf{J}_c$ to a matrix $\mathbf{J}'_c$ of equal rank which is drastically simpler, and calculating that $det(\mathbf{J}'_c) = 0$ on `MATHEMATICA` is a matter of less than a minute.

Moreover, we can explicitly compute the kernel of $\mathbf{J}'_c$, and thus by equivariance the kernel of $\mathbf{J}_c$. The result can be described as follows. Let

$$\begin{array}{lll} d_1 \otimes 1 := \langle a_{2,0}, a_{2,0}\rangle_{2,0} & e_1 \otimes 1 := \left\langle \langle a_{2,0}, \mathbf{b}\rangle_{1,0}, \mathbf{b}\right\rangle_{1,2} & b_{0,2} := \langle \mathbf{b}, \mathbf{b}\rangle_{1,1} \\ & & b_{2,0} := \langle \mathbf{b}, \mathbf{b}\rangle_{0,2} \\ d_2 \otimes 1 := \langle a_{0,2}, a_{0,2}\rangle_{0,2} & e_2 \otimes 1 := \left\langle \langle a_{0,2}, \mathbf{b}\rangle_{0,1}, \mathbf{b}\right\rangle_{1,2} & b_{2,4} := \langle \mathbf{b}, \mathbf{b}\rangle_{0,0} \end{array}$$

$$\begin{aligned} p_{2,0} &:= \left\langle \langle a_{2,0}, a_{0,2}\rangle_{0,0}, a_{0,2}\right\rangle_{0,2} \\ p_{2,4} &:= \left\langle \langle a_{2,0}, a_{2,0}\rangle_{0,0}, a_{2,0}\right\rangle_{2,0} \\ p_{0,2} &:= 16\left\langle \langle a_{2,0}, a_{0,2}\rangle_{0,0}, a_{2,0}\right\rangle_{2,0} + 9\left\langle \langle a_{0,2}, a_{0,2}\rangle_{0,0}, a_{0,2}\right\rangle_{0,2} \\ &\qquad -12\; c\; a_{0,2} + 3b_{0,2} \end{aligned}$$

Here, d_i and e_i are $\mathbb{C}$-valued functions on $\mathcal{V}$, while $b_{i,j}$ and $p_{i,j}$ are functions on $\mathcal{V}$ with values in $\mathcal{V}_{i,j}$. Then we have the

Proposition 4.2. *Let*

$$f_1^c := (4d_1 - 9d_2)(4d_1 + 27d_2 - 6c) + 72e_1 - 54e_2$$

and

$$f_2^c := 4d_2(4d_1 + 9d_2 - 3c)^2 + 96\,\langle p_{2,0}, b_{2,0}\rangle_{2,0} + 3\,\langle p_{0,2}, b_{0,2}\rangle_{0,2} + 48\,\langle p_{2,4}, b_{2,4}\rangle_{2,4}.$$

Then $d(K \circ f_i^c) = 0$ for $i = 1, 2$, and hence K maps F_0 into a level set of (f_1^c, f_2^c). Moreover, $rank(\mathbf{J}_c) = 10$ at $x \in \mathcal{V}$ if and only if $df_1^c \wedge df_2^c|_x \neq 0$.

Proof. The calculations involved to verify this Proposition were all performed on `MATHEMATICA` and will not be presented here in further detail. $\square$

We let

$$\Sigma_c := \{x \in \mathcal{V} \mid df_1^c \wedge df_2^c|_x = 0\}.$$

Then by Proposition 4.2. we know that $rank(K)_u = 10$ at $u \in F_0$ if and only if $K(u) \notin \Sigma_c$.

Let us define the functions $r^k_{i,j} : \mathcal{V} \to \mathcal{V}_{i,j}$ for $k = 1, 2$ by the equation

$$df_k^c = \tfrac{1}{6}\left\langle r^k_{2,0}, da_{2,0}\right\rangle_{2,0} + \tfrac{1}{2}\left\langle r^k_{0,2}, da_{0,2}\right\rangle_{0,2} + \tfrac{1}{2}\left\langle r^k_{1,2}, d\mathbf{b}\right\rangle_{1,2} \quad \text{for } k = 1, 2,$$

and define the vector fields Z_k, $k = 1, 2$, on F_0 by

$$\omega_0(Z_k) = r^k_{2,0} + r^k_{0,2} \quad \text{and} \quad \theta(Z_k) = r^k_{1,2}.$$

Then another `MATHEMATICA` calculation yields

Proposition 4.3. *The vector fileds Z_1 and Z_2 on F_0 defined above are symmetries, i.e. their Lie derivatives satisfy*

$$\mathfrak{L}_{Z_k}(\omega_0) = \omega_0 \quad \textit{and} \quad \mathfrak{L}_{Z_k}(\theta) = \theta \quad \textit{for } k = 1, 2. \tag{4-8}$$

Moreover, $[Z_1, Z_2] = 0$. $\square$

Corollary 4.4. *Either $rank(K) \equiv 10$ or $rank(K) < 10$ on all of F_0.*

Proof. From (4-8), standard arguments show that a symmetry either vanishes *everywhere* or *nowhere* on F_0. Thus, either Z_1 and Z_2 are pointwise linearly independent *everywhere* or *nowhere* on F_0.

From the definitions of the Z_k's it follows that Z_1 and Z_2 are linearly independent if and only if df_1^c and df_2^c are linearly independent. The claim follows then from Proposition 4.2. □

Definition 4.5. A torsion free $H_{1,2}$-connection is called *regular* if $rank(K) \equiv 10$, with the map $K : F_0 \to \mathcal{V}$ from above.

A torsion free $G_{1,2}$-connection is called *regular* if one and hence all of its associated $H_{1,2}$-structures are regular.

Thus, for a regular $H_{1,2}$-connection, the map K is a submersion onto an open subset of the regular part of a level set of (f_1^c, f_2^c) in $\mathcal{V}$.

Definition 4.6. Given constants $c, c_1, c_2 \in \mathbb{C}$ let

$$\mathcal{C}(c, c_1, c_2) := (f_1^c, f_2^c)^{-1}(c_1, c_2) \backslash \Sigma_c \subseteq \mathcal{V}.$$

If for a regular torsion free $H_{1,2}$-connection on $\pi : F_0 \to \mathcal{Z}$ the image of $K : F_0 \to \mathcal{V}$ is contained in $\mathcal{C}_{c,c_1,c_2}$ then we call the triple (c, c_1, c_2) the *structure constants of the connection.*

Let us now consider the question of *existence* of torsion free $H_{1,2}$-connections.

Theorem 4.7. *Given constants $c, c_1, c_2 \in \mathbb{C}$ let $\mathcal{C} := \mathcal{C}(c, c_1, c_2) \subseteq \mathcal{V}$. Then $\mathcal{C}$ can be covered by open sets U which have the following property: there exists a holomorphic principal $\mathbb{C}^2$-bundle $K : F_0 \to U$ over U and holomorphic 1-forms θ and ω_0 on F_0 with values in $\mathcal{V}_{1,2}$ and $\mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}$ respectively satisfying*

1. *the $\mathcal{V}_{1,2} \oplus \mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}$-valued 1-form $\theta + \omega_0$ is a coframe on F_0,*
2. *equations* (4-1) - (4-7) *are satisfied if we set $\omega_{0,0} = 0$, and if $K = \mathbf{a} + \mathbf{b}$ is the decomposition of K into its $\mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}$ and $\mathcal{V}_{1,2}$ components respectively.*

Moreover, the triple (F_0, θ, ω_0) is unique in the sense that if $(F_0', \theta', \omega_0')$ is another triple satisfying (1) *and* (2)*, then there is a bundle isomorphism between F_0 and F_0' which identifies the coframings.*

Proof. Let $\overline{a}, \overline{b}, \overline{r}_{i,j}^k$ and $\overline{J}$ be the restrictions of the functions $\mathbf{a}, \mathbf{b}, r_{i,j}^k$ and $\mathbf{J}_c$ respectively to $\mathcal{C}$.

By definition of $\mathcal{C}$ we have $rank(\overline{J}) \equiv 10$. From here it follows that there exist smooth 1-forms $\overline{\theta}$ and $\overline{\omega}_0$ on $\mathcal{C}$ with values in $\mathcal{V}_{1,2}$ and $\mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2}$ respectively such that

$$d\overline{a} + d\overline{b} = \overline{J}(\overline{\theta} + \overline{\omega}_0). \tag{4-9}$$

Since $d\overline{a}, d\overline{b}$ and $\overline{J}$ are holomorphic, we may assume that $\overline{\theta}$ and $\overline{\omega}_0$ are of type $(1,0)$.

The kernel of $\overline{J}$ is spanned at each point by the vectors $\overline{r}_{2,0}^k + \overline{r}_{0,2}^k + \overline{r}_{1,2}^k$ for $k = 1, 2$, so once one solution $(\overline{\theta}, \overline{\omega}_0)$ to (4-9) has been found, any other can be written in the form $(\overline{\theta} + \sum_k \overline{r}_{1,2}^k \alpha_k,\ \overline{\omega}_0 + \sum_k (\overline{r}_{2,0}^k + \overline{r}_{0,2}^k)\alpha_k)$ for unique 1-forms α_k, $k = 1, 2$.

Now we define the 2-forms

$$
\begin{aligned}
\overline{\Theta} &= d\overline{\theta} + \langle\!\langle \overline{\omega}_0, \overline{\theta} \rangle\!\rangle^{(1)} \\
\overline{\Phi} &= d\overline{\omega}_0 - \frac{1}{2}\left(\langle \overline{\omega}_{2,0}, \overline{\omega}_{2,0} \rangle_{1,0} + \langle \overline{\omega}_{0,2}, \overline{\omega}_{0,2} \rangle_{0,1} \right) - \overline{\Omega},
\end{aligned} \tag{4-10}
$$

where $\overline{\Omega}$ is given by replacing $a_{i,j}$ and θ by $\overline{a}_{i,j}$ and $\overline{\theta}$ respectively in (4-4).

After some calculation, the exterior derivative of (4-9) can be written in the form

$$0 = \overline{J}(\overline{\Theta} + \overline{\Phi}).$$

This implies that there are 2-forms $\overline{\Psi}_1$ and $\overline{\Psi}_2$ such that

$$\overline{\Theta} = \textstyle\sum_k \overline{r}^k_{1,2} \overline{\Psi}_k, \quad \text{and} \quad \overline{\Phi} = \textstyle\sum_k (\overline{r}^k_{2,0} + \overline{r}^k_{0,2}) \overline{\Psi}_k. \tag{4-11}$$

Substituting these relations into (4-10) and differentiating, we compute that

$$0 = \textstyle\sum_k \overline{r}^k_{1,2} d\overline{\Psi}_k, \quad \text{and} \quad 0 = \textstyle\sum_k (\overline{r}^k_{2,0} + \overline{r}^k_{0,2}) d\overline{\Psi}_k.$$

Since the functions $\overline{r}^k_{1,2} + \overline{r}^k_{2,0} + \overline{r}^k_{0,2}$ are linearly independent, we conclude that

$$d\overline{\Psi}_k = 0 \quad \text{for } k = 1, 2.$$

Let $U \subset \mathcal{C}$ be an open set on which the $\overline{\Psi}_k$'s are *exact*. Clearly, $\mathcal{C}$ can be covered by such open sets. Let α_k be 1-forms on U such that $d\alpha_k = \overline{\Psi}_k$. If we replace the pair $(\overline{\theta}, \overline{\omega}_0)$ by $\left(\overline{\theta} - \sum_k \overline{r}^k_{1,2} \alpha_k,\ \overline{\omega}_0 - \sum_k \left(\overline{r}^k_{2,0} + \overline{r}^k_{0,2}\right) \alpha_k\right)$, then another calculation shows that for this new pair

$$
\begin{aligned}
0 &= d\overline{\theta} + \langle\!\langle \overline{\omega}_0, \overline{\theta} \rangle\!\rangle^{(1)} \\
0 &= d\overline{\omega}_0 - \frac{1}{2}\left(\langle \overline{\omega}_{2,0}, \overline{\omega}_{2,0} \rangle_{1,0} + \langle \overline{\omega}_{0,2}, \overline{\omega}_{0,2} \rangle_{0,1} \right) - \overline{\Omega}.
\end{aligned} \tag{4-12}
$$

Note that from (4-10) and (4-11) it follows that $\overline{\Psi}_k$ has no $(0, 2)$-part. Thus, α_k can be chosen to be of type $(1, 0)$. But then (4-12) implies that $\overline{\theta}$ and $\overline{\omega}_0$ are holomorphic 1-forms.

Now we let $F_0 := U \times \mathbb{C}^2$ with coordinates $(\overline{a}, \overline{b}, s_1, s_2)$ and define the 1-forms

$$(\theta, \omega_0) := \left(\overline{\theta} + \sum_k \overline{r}^k_{1,2} ds_k,\ \overline{\omega}_0 + \sum_k \left(\overline{r}^k_{2,0} + \overline{r}^k_{0,2}\right) ds_k \right).$$

Then it is not hard to show that (θ, ω_0) is a holomorphic coframe on F_0 satisfying the postulates of the Theorem.

The uniqueness of (θ, ω_0) follows from the standard facts about mappings preserving coframings [G]. $\square$

We are now ready to prove the existence result for H connections.

Corollary 4.8. *For any constants $c, c_1, c_2 \in \mathbb{C}$ and any point $u \in \mathcal{C}(c, c_1, c_2)$, there exists a regular torsion free connection on some $H_{1,2}$-structure $\pi : F_0 \to \mathcal{Z}$ where $\mathcal{Z}$ is some sixfold so that the image of the curvature map $K : F_0 \to \mathcal{C}_{c,c_1,c_2}$ contains u.*

Proof. Let $U \subseteq \mathcal{C}_{c,c_1,c_2}$ be an open neighborhood of u for which the conclusion of Theorem 4.7. holds, i.e. there is a principal $\mathbb{C}^2$-bundle $K : \tilde{F}_0 \to U$ and a coframe $(\tilde{\theta}, \tilde{\omega})$ on $\tilde{F}_0$ satisfying (4-1) - (4-7) with $\omega_{0,0} = 0$.

Pick a point $v \in \tilde{F}_0$ with $K(v) = u$. Since by the structure equations we have $d\tilde{\theta} \equiv 0 \mod \tilde{\theta}$, it follows that $\tilde{F}_0$ is foliated by integral leafs on which $\tilde{\theta}$ vanishes. For some sufficiently small neighborhood V of v, there exists a submersion $\pi : V \to \mathcal{Z}$ onto some sixfold $\mathcal{Z}$ such that $\ker(\pi_*) = \tilde{\theta}^{\perp}$.

Moreover, standard arguments show that there is an inclusion $\imath : V \hookrightarrow F_0 \subseteq \mathfrak{F}$ of V into an $H_{1,2}$-structure F_0 on $\mathcal{Z}$ such that $\imath^*(\theta) = \tilde{\theta}$ where θ denotes the tautological form on F_0.

Also, there is an unique $H_{1,2}$-connection ω_0 on F_0 with $\imath^*(\omega_0) = \tilde{\omega}_0$. From the structure equations it is then evident that the curvature map $K : F_0 \to \mathcal{V}$ satisfies $u \in K(F_0) \subseteq \mathcal{C}_{c,c_1,c_2}$. $\square$

As a consequence of the proof of Theorem 4.7. we have

Corollary 4.9. *All regular torsion free $H_{1,2}$- and $G_{1,2}$-connections are holomorphic.* $\square$

§5 Summary.

In §3 we have shown that the moduli space $\mathcal{Z}$ of rational curves C in a complex threefold $\mathcal{W}$ whose normal bundle $N_C \to C$ is equivalent to $\mathcal{O}(2) \oplus \mathcal{O}(2)$ forms a six dimensional manifold which carries a natural $G_{1,2}$-structure $\pi : F \to \mathcal{Z}$. By Theorem 3.11. *most* of the intrinsic torsion of this structure vanishes. A natural question is whether every holomorphic $G_{1,2}$-structure whose torsion is of the form of Theorem 3.11. arises from such a moduli space.

The answer is negative in general. The reason is that Proposition 2.9. gives some first order restriction which is not automatically satisfied if the torsion is of the form of Theorem 3.11., not even when the torsion vanishes.

Before stating this result, let us write out the decompositions

$$\begin{array}{rclcccccc} \theta & = & \theta_{1,2} & x \otimes x^2 & + & \theta_{1,0} & x \otimes xy & + & \theta_{1,-2} x \otimes y^2 \\ & + & \theta_{-1,2} & y \otimes x^2 & + & \theta_{-1,0} & y \otimes xy & + & \theta_{-1,-2} y \otimes y^2 \end{array}$$

and

$$\begin{array}{rclcccccccc} \omega & = & \omega_{0,0} & 1 \otimes 1 & & & & & & \\ & + & \omega^{02}_{0,2} & 1 \otimes x^2 & + & \omega^{02}_{0,0} & 1 \otimes xy & + & \omega^{02}_{0,-2} & 1 \otimes y^2 \\ & + & \omega^{20}_{2,0} & x^2 \otimes 1 & + & \omega^{20}_{0,0} & xy \otimes 1 & + & \omega^{20}_{-2,0} & y^2 \otimes 1. \end{array}$$

Proposition 5.1. *Suppose $\mathcal{Z}$ is the moduli space of rational curves in the threefold $\mathcal{W}$ whose normal bundle is equivalent to $\mathcal{O}(2) \oplus \mathcal{O}(2)$, and suppose furthermore that the associated $G_{1,2}$-structure $\pi : F \to \mathcal{Z}$ is torsion free. Then the torsion free connection on F is locally symmetric.*

Proof. By Theorem 3.11. the torsion free connection on $\pi : F \to \mathcal{Z}$ must be special. Thus, by Proposition 2.9.,

Of course, this means that $\mathcal{I}$ must satisfy the *Frobenius condition* $d\mathcal{I} \equiv 0 \mod \mathcal{I}$. However, from the structure equations (4-1) - (4-4) we compute that $d\omega^{02}_{0,-2} \equiv 9\left\langle a_{0,2}, x^2\right\rangle_2 \theta_{1,0} \wedge \theta_{-1,0} \mod \mathcal{I}$. From here it follows that the Frobenius condition is satisfied if and only if $a_{0,2} \equiv 0$ on F. By (4-5) and (4-6), this implies that $\mathbf{b} \equiv 0$ and $\langle a_{2,0}, a_{2,0}\rangle_{2,0} = \frac{3}{4}c$.

However, $\mathbf{b}$ represents the covariant derivative of the curvature tensor. It follows that F is locally symmetric. □

Let us now consider the question which regular torsion free $G_{1,2}$-connections $(\pi : F \to \mathcal{Z}, \theta, \omega)$ admit a restriction $(\pi : \hat{F} \to \mathcal{Y}, \hat{\theta}, \hat{\omega})$ in the sense of Definition 3.6. Since the holonomy of a torsion free $G_{1,2}$-connection is contained in $H_{1,2}$ it follows that the holonomy of the restriction to $\mathcal{Y}$ is contained in H_3.

Proposition 5.2. *Let (θ, ω) be a regular torsion free $H_{1,2}$-connection on $\pi : F_0 \to \mathcal{Z}$ with structure constants (c, c_1, c_2). Then F_0 admits a restriction to an H_3-connection $\pi : \hat{F}_0 \to \mathcal{Y}$ if and only if $c_1 = 0$.*

In this case, the restriction $\hat{F}_0$ is uniquely determined, and the connection on $\pi : \hat{F}_0 \to \mathcal{Y}$ is regular in the sense of [Br].

Conversely, given a regular H_3-connection on $\pi : \hat{F}_0 \to \mathcal{Y}$, there is a unique regular torsion free $H_{1,2}$-connection which extends the connection on $\hat{F}_0$.

Proof. Let $\pi : F_0 \to \mathcal{Z}$ be the torsion free regular $H_{1,2}$-connection. If a restriction on $\hat{F}_0 \to \mathcal{Y}$ exists then $T\hat{F}_0$ must be annihilated by the ideal

$$\mathcal{J} = \{\theta_{-1,0} - 2\theta_{1,-2},\ \theta_{1,0} - 2\theta_{-1,2},\ \omega^{02}_{0,i} - \omega^{20}_{i,0}, i = 0, 1, 2\}.$$

Thus, $\mathcal{J}$ must satisfy the *Frobenius condition* $d\mathcal{J} \equiv 0 \mod \mathcal{J}$. A calculation using the structure equations (4-1) - (4-4) yields that this is the case if and only if

$$2a_{2,0} = 3a_{0,2}. \tag{5-1}$$

Taking the exterior derivative $2da_{2,0} - 3da_{0,2} \mod \mathcal{J}$, we conclude that $\mathbf{b}$ must be of the form

$$\mathbf{b} = x \otimes b^3_x + y \otimes b^3_y \tag{5-2}$$

for some $\mathcal{V}_3$-valued function b^3 where the subscripts stand for partial derivatives.

Let us define $\hat{F}_0 \subseteq F_0$ by (5-1) and (5-2). Then it is evident that any reduction of F_0 must be contained in $\hat{F}_0$.

From the structure equations (4-5) - (4-7) we calculate that the differentials of the components of (5-1) and (5-2) are linearly independent. Also, substituting (5-1) and (5-2) into f^c_1 from Proposition 4.2. we calculate $f^c_1 = 0$.

Therefore, $\hat{F}_0 = \emptyset$ if $c_1 \neq 0$. Conversely, if $c_1 = 0$ one can verify that $\hat{F}_0$ is non-empty and hence an eight dimensional analytic submanifold of F_0. Moreover, $\dim(T\hat{F}_0 \cap \ker \pi_*) \equiv 4$, and so $\mathcal{Y} := \pi(\hat{F}_0)$ is an analytic submainfold of $\mathcal{Z}$. Now it is easy to verify that $(\pi : \hat{F}_0 \to \mathcal{Y}, \hat{\theta}, \hat{\omega})$ with $\hat{\theta}$ and $\hat{\omega}$ as in Definition 3.6. is the desired restriction. Of course, (5-1) and (5-2) determine $\hat{F}_0$ uniquely.

Note that this restriction is a torsion free H_3-connection. The final statement follows from the classification of regular H_3-connections in [Br]. They are uniquely determined by two constant parameters, and it left to the reader to verify that these correspond to the constants c and c_2. □

Remark.

(1) It seems likely that the last statement in Proposition 5.2. holds true even if the H_3-connection on $\mathcal{Y}$ is *not* regular, i.e. in this case there should still be an extension to a unique torsion free $H_{1,2}$-connection. There does not seem to be any substantial obstacle to proving this other than the immense calculations required to determine the non-regular $H_{1,2}$-connections.

(2) Since every torsion free holomorphic G_3-connection is equivalent to the moduli space of contact curves in a contact threefold $\mathcal{W}$ [Br], it follows from the results in §3 that *every holomorphic torsion free G_3-connection can be extended to a $G_{1,2}$-connection whose torsion is given as in the proof of* Theorem 3.13.

A characterization of H_3-connections is therefore that they are precisely those G_3-connections which can be extended to a *torsion free* $G_{1,2}$-connection. (cf. Theorem 0.4.)

(3) A somewhat surprising aspect comes from a comparison of Proposition 5.1. and Proposition 5.2. Namely, if $\mathcal{Y}$ admits a regular torsion free H_3-connection then on the one hand, by (2), the connection on $\mathcal{Y}$ can be extended to a connection on the moduli space $\mathcal{Z}$ of rational curves in $\mathcal{W}$.

On the other hand, if we let $\mathcal{Z}'$ be the *torsion free* extension from Proposition 5.2. then it follows from Proposition 5.1. that $\mathcal{Z}'$ is different from $\mathcal{Z}$ unless both $\mathcal{Z}$ and $\mathcal{Y}$ are flat: indeed, the only locally symmetric H_3-connection is the flat one. (cf. Theorem 0.1.)

In other words, the extension $\mathcal{Z}$ of $\mathcal{Y}$ which seems most natural in the geometric sense is different from the extension $\mathcal{Z}'$ of $\mathcal{Y}$ which is most natural from the torsion point of view.

Definition 5.3. Let $\mathcal{P}$ be a complex five dimensional manifold. A *linear rank 2 Pfaffian system on* $\mathcal{P}$ or, for short, a *Pfaffian structure* on $\mathcal{P}$ is a differential ideal $\mathcal{I}$ on $\mathcal{P}$ with the property that, locally, there is a holomorphic coframe $\kappa_1, \kappa_2, \alpha, \beta_1, \beta_2$ on $\mathcal{P}$ such that

$$\mathcal{I} = \{\kappa_1, \kappa_2\}$$

and

$$d\kappa_i = \alpha \wedge \beta_i \mod \mathcal{I}, \quad i = 1, 2.$$

A curve $C \subseteq \mathcal{P}$ is called an *integral curve* if the tangent vectors of C are anihilated by $\mathcal{I}$.

A Pfaffian structure on $\mathcal{P}$ may also be regarded as a rank 2 subbundle $L \subseteq T^*\mathcal{P}$ where L is locally spanned by κ_1 and κ_2.

For example, if $\mathcal{W}$ is any three dimensional manifold then $\mathcal{P} := \mathbb{P}T\mathcal{W}$ carries a canonical Pfaffian structure [EDS]. Namely, for local coordinates (x, y, z) and (x, y, z, u, v) on $\mathcal{W}$ and $\mathcal{P}$ respectively such that the bundle map $\pi : \mathcal{P} \to \mathcal{W}$ is given by $(x, y, z, u, v) \mapsto (x, y, z)$, this system is given as $\mathcal{I} := \{dy - u dx, dz - v dx\}$.

Thus, for a curve of the form $(x, y(x), z(x))$ in $\mathcal{W}$ there is a unique integral lift to $\mathcal{P}$, namely $(x, y(x), z(x), y'(x), z'(x))$.

A key observation is now given by the following

Theorem 5.4. *Let $\pi : F \to \mathcal{Z}$ be a holomorphic $G_{1,2}$-connection whose torsion is of the form of Theorem 3.11. Then, locally, $\mathcal{Z}$ is (contained in) the moduli space of all integral curves C of a Pfaffian structure $\mathcal{I}$ on some fivefold $\mathcal{P}$.*

Proof. The proof requires to compute the structure equations for connections whose torsion is of the required form. Since these equations are quite complex and since we shall not need them any further, they will be omitted.

It follows from these equations that the differential ideal

$$\mathcal{J} := \{\theta_{1,0}, \theta_{-1,0}, \theta_{1,-2}, \theta_{-1,-2}, \omega^{02}_{0,-2}\}$$

does satisfy the Frobenius condition $d\mathcal{J} \equiv 0 \mod \mathcal{J}$. Thus, at least locally, there is a map $p : F \to \mathcal{P}$ onto some five dimensional complex manifold $\mathcal{P}$ such that $\ker(p_*) = \mathcal{J}^\perp$.

For each point $t \in \mathcal{Z}$, we let $C_t := p(\pi^{-1}(t))$. It is then easy to see that C_t is a rational curve in $\mathcal{P}$, and hence we may regard $\mathcal{Z}$ as the moduli space of certain rational curves in $\mathcal{P}$.

Let $\mathcal{I}_0 := \{\theta_{1,-2}, \theta_{-1,-2}\}$. Then a calculation shows that for each vector field $X \in \mathcal{J}^\perp$ on F, $\mathcal{L}_X(\mathcal{I}_0) \subseteq \mathcal{I}_0 \mod \mathcal{J}$. Therefore, there is a differential system $\mathcal{I}$ on $\mathcal{P}$ such that $p^*(\mathcal{I}) = \mathcal{I}_0$.

Taking the exterior derivatives of $\theta_{1,-2}$ and $\theta_{-1,-2}$ it follows that $\mathcal{I}$ is indeed a Pfaffian structure on $\mathcal{P}$. Moreover, since $\pi^{-1}(t)$ is integral to $\mathcal{I}_0$ for all $t \in \mathcal{Z}$ it follows that C_t is an integral curve for all $t \in \mathcal{Z}$. $\square$

Theorem 5.4. suggests that it should be more natural to regard G_3-structures, H_3-structures and $G_{1,2}$-structures as moduli spaces of integral curves of a fivefold with Pfaffian structure rather than as curves in a threefold. Indeed, the remarks preceding Theorem 5.4. indicate how the moduli space of curves in a threefold $\mathcal{W}$ may be regarded merely as a special case of this.

It should also be instructive to see how the local invariants of a Pfaffian structure on $\mathcal{P}$ [C] relate to the associated $G_{1,2}$-structure. This will be pursued in a sequel of the present paper.

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